# A Geometric Treatment of Generalized Inverses and Semigroups of Nonnegative Matrices

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ABSTRACT

The purpose of this paper is to provide a unified treatment from the geometric viewpoint of the following closely related aspects of nonnegative matrices: nonnegative matrices with nonnegative generalized inverses of various kinds; nonnegative rank factorization; regular elements, Green's relations, and maximal subgroups of the semigroups of nonnegative matrices, stochastic matrices, column stochastic matrices, and doubly stochastic matrices.

## 1. INTRODUCTION

Whereas a lot of work has been done on nonnegative matrices with different kinds of nonnegative generalized inverses and on the algebraic structure of various semigroups of nonnegative matrices (see the references at the end), only a few treatments of the above topics have been given from the geometrical viewpoint (for instance, Flor [19] and Smith [51]). This paper is written to support the view that the use of geometrical methods in the study of nonnegative matrices can be fruitful and deserves better attention. At least, there is the evidence that Flor's characterization of nonnegative idempotent matrices, which has been fundamental to much previous work, was derived by a geometrical method.

The operator-theoretic viewpoint will be exploited. An $m\times n$ nonnegative matrix is often looked upon as a linear operator which sends the cone $R^n_+$ into the cone $R^m_+$, where $R^n_+$ denotes the nonnegative orthant of $R^n$. For a nonnegative matrix with various kinds of nonnegative generalized inverses, its behavior as a linear operator is completely characterized by the results of Tam

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[52]. For instance, if $A$ is an $n\times n$ nonnegative matrix with a nonnegative group inverse, then $A$ can be realized by: first, a nonnegative projection on $\operatorname{Im} A$, the image space of $A$, and then a nonsingular linear mapping on $\operatorname{Im} A$, which is determined by a permutation of the extreme rays of the simplicial cone $\operatorname{Im} A\cap R^n_+$. (A cone is *simplicial* iff it is generated by linearly independent vectors.) This observation enables us to provide, in Section 3 among others, geometrical proofs of the known characterizations of maximal subgroups of nonnegative matrices, stochastic matrices, and column stochastic matrices (Flor [19], Plemmons [39], Schwarz [47], Wall [56]).

In the study of generalized inverses of nonnegative matrices, the nonnegative rank factorization has been a useful tool to many previous research workers. Although this tool does not play an important role in our approach, it does have some connection. In Section 4 two equivalent conditions for an $m\times n$ nonnegative matrix $A$ of rank $r$ to have a nonnegative rank factorization are given. One condition is that there exists a simplicial cone $K$ such that $AR^n_+\subseteq K\subseteq \operatorname{Im} A\cap R^m_+$. The condition has appeared, in a slightly different form, in a recent paper of Campbell and Poole [14]. The other condition, which is fairly obvious but is important, says that $A$ can be represented as a sum of $r$ nonnegative rank-one matrices. In fact, each such representation of $A$ corresponds to a nonnegative rank factorization of $A$ (which is unique in certain sense), and vice versa. From this condition and the results of Tam [52] we obtain Theorem 4.2, which is fundamental to our treatment of the various topics concerning nonnegative matrices with nonnegative (1)-inverses.

In Section 5 we give several equivalent conditions for an $m\times n$ nonnegative matrix $A$ of rank $r$ to have various kinds of nonnegative generalized inverses. One type of conditions is expressed in terms of the extreme vectors of the simplicial cones $\operatorname{Im} A\cap R^m_+$ and $\operatorname{Im} A^T\cap R^n_+$, which appear in the "unique" representation of $A$ as a sum of $r$ nonnegative rank-one matrices (see Theorem 4.2). From this type of conditions the various known or partially known characterizations of $A$ in matrix block forms are derived. In contrast with the usual matrix-computational methods, our method appears to be simpler and more direct. As a by-product of our approach, we also obtain results about the geometry of the nonnegative orthant, which are interesting by themselves and may be useful elsewhere. (See also our Section 8.)

Besides the operator-theoretic viewpoint, another useful geometrical idea is to consider the polyhedral cone $G(A)$ [or $G(A^T)$] generated by the column [or row] vectors of a matrix A. It is easy to see that the matrix equation $A=BX$, where $X$ is a nonnegative matrix, can be interpreted as a statement about containment between cones: $G(A)\subseteq G(B)$. It follows that for $A$, $B\in N_n$, the semigroup of $n\times n$ nonnegative matrices, $A\mathcal{R} B$ iff $G(A)=G(B)$. The Green's relations on $N_n$ have been studied before by Hartfiel, Maxson, and Plemmons [22]. However, the above simple, yet useful, characterization of the $\mathcal{R}$ relation and the corresponding characterization of the $\mathcal{L}$ relation cannot be found in their paper. Besides the $\mathcal{R}$ and $\mathcal{L}$ relations, in Section 6 we also provide geometric characterizations of the $\mathcal{D}$ and $\mathcal{J}$ relations of $N_n$. A characterization of the $\mathcal{J}$ relation was not given in [22].

Using our method, in Section 7 we study the algebraic properties of the semigroups of stochastic matrices, column stochastic matrices, and doubly stochastic matrices. A geometrical proof of a well-known characterization of maximal subgroups of doubly stochastic matrices is given. For $S_n$, the semigroup of $n\times n$ stochastic matrices, we characterize its regular elements and the $\mathcal{R}$ relation for its regular elements. These topics have been treated before by Wall [55] and [56]. Our approach, however, yields more transparent results, which contain Wall's results as special cases. For instance, for any $A\in S_n$, $A$ is regular in $S_n$ iff $A$ is regular in $N_n$ and each column of $A$ is either the zero vector or an extreme vector of the cone $G(A)$. Also, for regular elements $A$, $B$ of $S_n$, $A\mathcal{R}B$ in $S_n$ iff $A\mathcal{R}B$ in $N_n$. (The corresponding result for the $\mathcal{L}$ relation holds trivially without the regularity assumption on $A$ and $B$.) Nevertheless, the characterization of the $\mathcal{R}$ relation of $S_n$ is still an unsolved problem. The problem is somehow related to an old (wrong) conjecture of Kakutani concerning doubly stochastic matrices. Some related results will be given.

In Section 8, we end with remarks and open questions.

## 2. PRELIMINARIES

A nonempty subset $K$ in $R^n$ (the vector space of all $n$-dimensional real column vectors) is called a *(convex) cone* if $K$ is closed under addition and multiplication by nonnegative scalars: $K$ is *pointed* if $K\cap(-K)=\{0\}$; $K$ is *solid* if $\operatorname{int}K\neq\varnothing$, or equivalently, $K-K=R^n$. If $K$ is topologically closed and satisfies all of the above properties, $K$ is called a *proper cone*. A nonempty subset $F$ of a closed, pointed cone $K$ is called a *face* of $K$, denoted by $F\trianglelefteq K$, if $F$ is itself a cone and in addition satisfies the following: if $x,y\in K$ such that $x+y\in F$, then $x,y\in F$. If $S\subseteq K$, then the smallest face of $K$ containing $S$ is called the face generated by $S$ and is denoted by $\Phi(S)$. If $S=\{x\}$, we write $\Phi(x)$ for simplicity. If $x\neq0$ and if $\Phi(x)=\{\alpha x:\alpha\geqslant0\}$, then $\Phi(x)$ is called an *extreme ray* and $x$ an *extreme vector* of $K$.

For any subset $S$ of $R^n$, the set $S^*=\{z\in R^n:(z,y)\geqslant0 \text{ for all } y\in S\}$ is called the *dual* of $S$. [The inner product of $R^n$ is the usual one: $(z,y)=z^Ty$.] The dual $K^*$ of a proper cone $K$ is also a proper cone known as the *dual cone* of $K$.

We shall use the words "matrix" and "linear operator" interchangeably. The linear span of a set $S$ will be denoted by span $S$. If $K_1$ is a cone in $R^n$ and

$K_2$ a cone in $R^m$, then a linear operator in $\mathrm{Hom}(R^n, R^m)$ which maps $K_1$ into $K_2$ is called a *positive operator from* $K_1$ *to* $K_2$. The set of all such positive operators is denoted by $\pi(K_1, K_2)$. For simplicity, we write $\pi(K)$ for $\pi(K, K)$. Obviously, if $A$ is an $m\times n$ nonnegative matrix, then $A\in\pi(R^n_+, R^m_+)$.

Only real matrices will be considered. Let $A$ be an $m\times n$ matrix. Consider the following matrix equations.

$$
\begin{array}{rlrl}
AXA=A & (1) & XAX=X & (2)\\
(AX)^T=AX & (3) & (XA)^T=XA & (4)\\
AX=XA & (5) & A^k=XA^{k+1} & (1^k)
\end{array}
$$

Any solution $X$ of (1) is called a (1)*-inverse* of $A$ and is usually denoted by $A^{(1)}$. A (1)-inverse which also satisfies (2) is called a *semiinverse* of $A$. There is always a unique solution to Equations (1), (2), (3), and (4). It is called the *Moore-Penrose inverse* of $A$ and is denoted by $A^+$. The *group inverse* of $A$, if it exists, is the unique matrix $A^\#$ which satisfies (1), (2), and (5). (Then $A$ is a square matrix.) It can be shown that $A^\#$ exists iff $A$ is contained in a group of matrices. Furthermore, the inverse of $A$ in any such group is always $A^\#$. When $A$ is a square matrix, there is a unique solution to Equations (2), (5), and $(1^k)$, where $k$ is the index of $A$, i.e. the smallest positive integer $k$ such that $\mathrm{rank}\, A^k=\mathrm{rank}\, A^{k+1}$. The solution is called the *Drazin pseudoinverse* of $A$ and is denoted by $A^{(d)}$. For the geometrical properties of these generalized inverses, we refer the readers to Section 2 of the author's previous paper [52]. In that paper the author has obtained characterizations of linear operators in $\pi(K_1, K_2)$ which have various kinds of generalized inverses in $\pi(K_2, K_1)$, as well as some results on nonnegative idempotent matrices. For convenience, we collect these results below. Hereafter in this section, $K_1$ (or $K$) is a proper cone of $R^n$, and $K_2$ a proper cone of $R^m$.

THEOREM 2.1. *Let $A\in\pi(K_1, K_2)$. A necessary and sufficient condition for the existence of an $A^{(1)}$ in $\pi(K_2, K_1)$ is: there exists a subspace $H$ of $R^n$ such that* $\mathrm{span}(H\cap K_1)=H$, *$A$ takes $H\cap K_1$ one-to-one onto $\mathrm{Im}\, A\cap K_2$, and there is a projection in $\pi(K_2)$ with the same image space as $A$. When this condition is satisfied, there is also a projection $P$ in $\pi(K_1)$ with image space $H$ such that $A=AP$.*

THEOREM 2.2. *Let $A\in\pi(K)$. The group inverse $A^\#$ of $A$ exists and belongs to $\pi(K)$ iff* $A(\mathrm{Im}\, A\cap K)=\mathrm{Im}\, A\cap K$.

THEOREM 2.3. *Let $A\in\pi(K)$. The Drazin pseudoinverse $A^{(d)}$ of $A$ belongs to $\pi(K)$ iff* $A(\mathrm{Im}\, A^k\cap K)=\mathrm{Im}\, A^k\cap K$, *where $k$ is the index of $A$.*

THEOREM 2.4. *Let $A \in \pi(K_1, K_2)$. The Moore-Penrose inverse $A^+$ of $A$ belongs to $\pi(K_2, K_1)$ iff the orthogonal projection of $R^n$ on* $\operatorname{Im} A^+$ *belongs to $\pi(K_1)$, the orthogonal projection of $R^m$ on* $\operatorname{Im} A$ *belongs to $\pi(K_2)$, and $A$ takes the cone* $\operatorname{Im} A^+ \cap K_1$ *one-to-one onto the cone* $\operatorname{Im} A \cap K_2$.

PROPOSITION 2.5. *Let $H$ be a subspace of $R^n$ such that* $\operatorname{span}(H \cap R^n_+) = H$. *There exists a nonnegative idempotent matrix with image space $H$ iff $H \cap R^n_+$ is a simplicial cone.*

PROPOSITION 2.6. *An $n \times n$ nonnegative matrix of rank $r$ is idempotent iff it is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$ where $x_i, y_i \in R^n_+$ and $y_i^T x_j = \delta_{ij}$.*

The lemma below can be established by Lemma 4.3 of our previous paper [52], but we shall supply a more direct and constructive proof.

LEMMA 2.7. *Let $A$ be an $m \times n$ real matrix. Suppose that the cone $AR^n_+$ is contained in a simplicial cone with extreme vectors $x_1, \dots x_r$. Then there exist unique vectors $y_1, \dots, y_r \in R^n_+$ such that $A = x_1 y_1^T + \cdots + x_r y_r^T$.*

*Proof.* Denote by $a_i$ the $i$th column vector of $A$. Then there exist unique nonnegative scalars $\lambda_{ij}$ such that $a_i = \lambda_{i1} x_1 + \cdots + \lambda_{ir} x_r$, whence

$$\begin{aligned} A &= [a_1, \dots, a_n] \\ &= [\lambda_{11} x_1 + \cdots + \lambda_{1r} x_r, \dots, \lambda_{n1} x_1 + \cdots + \lambda_{nr} x_r] \\ &= x_1 y_1^T + \cdots + x_r y_r^T, \end{aligned}$$

where $y_i = (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni})^T \in R^n_+$. The uniqueness of the scalars $\lambda_{ij}$ guarantees the uniqueness of the vectors $y_i$. ∎

From the operator-theoretic viewpoint, the behavior of a nonnegative matrix with various kinds of nonnegative generalized inverses is understood well from the above results. For instance, let $A$ be an $m \times n$ nonnegative matrix of rank $r$ which has a nonnegative (1)-inverse. According to Theorem 2.1, there exists an $r$-dimensional subspace $H$ of $R^n$ such that $A$ takes the cone $H \cap R^n_+$ isomorphically onto the cone $\operatorname{Im} A \cap R^m_+$. This necessarily implies that $AR^n_+ = \operatorname{Im} A \cap R^m_+$. Furthermore, since there exists a nonnegative idempotent matrix with the same image space as $A$, by Proposition 2.5 the cone $\operatorname{Im} A \cap R^m_+$ is simplicial. Clearly $A^T$ also has a nonnegative (1)-inverse, and likewise $\operatorname{Im} A^T \cap R^n_+$ is a simplicial cone. However, the fact that both $\operatorname{Im} A \cap R^m_+$ and $\operatorname{Im} A^T \cap R^n_+$ are simplicial cones does not imply the existence of a nonnegative (1)-inverse of $A$. For instance, let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A$ is a symmetric nonnegative matrix of rank 2. $\operatorname{Im} A \cap R^3_+ = \operatorname{Im} A^T \cap R^3_+$ is the simplicial cone generated by the vectors $(1,0,0)^T$ and $(0,1,0)^T$. But $A$ has no nonnegative (1)-inverse, because $AR^3_+ \neq \operatorname{Im} A \cap R^3_+$. Nevertheless, we have the following.

Proposition 2.8. *Let $A$ be an $m \times n$ nonnegative matrix such that $AR^n_+ = \operatorname{Im} A \cap R^m_+$. Then the following are equivalent:*

(i) *$A$ has a nonnegative (1)-inverse.*
(ii) *$AR^n_+$ is a simplicial cone.*
(iii) *$A^T R^m_+$ is a simplicial cone.*

*Proof.* (i)$\Rightarrow$(ii): From the above discussion, $AR^n_+ = \operatorname{Im} A \cap R^m_+$ and the cone $AR^n_+$ is simplicial.

(ii)$\Rightarrow$(i): Let rank $A = r$. Being generated by the column vectors of $A$, the cone $AR^n_+$ is obviously of dimension $r$. (By the dimension of a cone, we mean the dimension of its linear span.) Denote by $a_i$ the $i$th column of $A$, and by $e_i$ the vector in $R^n$ with 1 in the $i$th position and 0 elsewhere. Let $a_{i_1}, \ldots, a_{i_r}$ be the extreme vectors of the simplicial cone $AR^n_+$. As $Ae_{i_k} = a_{i_k}$, $1 \leqslant k \leqslant r$, $A$ maps $\operatorname{span}\{e_{i_1}, \ldots, e_{i_r}\} \cap R^n_+$ one-to-one onto $AR^n_+$, which is $\operatorname{Im} A \cap R^m_+$. Hence, by Theorem 2.1, $A$ has a nonnegative (1)-inverse.

By Proposition 5.1 of Tam [52], the given assumption $AR^n_+ = \operatorname{Im} A \cap R^m_+$ implies that $A^T R^m_+ = \operatorname{cl} A^T R^m_+ = \operatorname{Im} A^T \cap R^m_+$. The equivalence of (i) and (iii) now follows from the "dual" statement of (i)$\Leftrightarrow$(ii), as $A$ has a nonnegative (1)-inverse iff $A^T$ has. ■

The following well-known characterization of nonnegative idempotent matrices was first proved by Flor [19] from the operator-theoretic viewpoint. The result has been fundamental to the matrix-computational methods. Here we give an alternative proof using Proposition 2.6.

THEOREM 2.9. *An $n\times n$ nonnegative matrix $E$ of rank $r$ is idempotent iff there exists a permutation $P$ such that*

$$PEP^T=\begin{bmatrix} J & JD & 0 & 0\\ 0 & 0 & 0 & 0\\ CJ & CJD & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

*where $J$ is a direct sum of positive rank-one idempotent matrices, $C, D$ are nonnegative matrices of appropriate sizes, and the zero blocks along the main diagonal are square.*

*Proof.* "If" part: Straightforward verification.

"Only if" part: By Proposition 2.6, $E$ can be expressed as

$$x_1y_1^T+\cdots+x_ry_r^T,$$

where $x_i$ and $y_i$ are nonnegative vectors of $R^n$ and $y_i^Tx_j=\delta_{ij}$. Clearly $x_i$ and $y_j$ have nonzero entries at some common positions iff $i=j$. Denote by $x_{ij}$ the $j$th component of $x_i$. Similar meanings hold for the symbol $y_{ij}$. Let $N=\{1,2,\dots,n\}$, and let

$$M_i=\{k\in N: x_{ik}\neq 0 \text{ and } y_{ik}\neq 0\},\qquad 1\leqslant i\leqslant r,$$

$$M'=\{k\in N: x_{ik}=0 \text{ for all } i \text{ but } y_{jk}\neq 0 \text{ for some } j\},$$

$$M''=\{k\in N: x_{ik}\neq 0 \text{ for some } i \text{ but } y_{ik}=0 \text{ for all } i\},$$

$$M'''=\{k\in N: x_{ik}=y_{ik}=0 \text{ for all } i\}.$$

Obviously, $N=\bigcup_{i=1}^{r}M_i\cup M'\cup M''\cup M'''$. Permute the entries of $x_i$ and $y_i$ so that they can be partitioned into blocks with entry indices corresponding to $M_1, M_2,\dots, M_r$, $M'$, $M''$, and $M'''$, and in that order. Let $P$ be the corresponding permutation matrix. Then

$$(Px_i)^T=\left[0 \quad \cdots \quad u_i^T \quad \cdots \quad 0 \,\middle|\, 0 \,\middle|\, c_i^T \,\middle|\, 0\right],$$

$$(Py_i)^T=\left[0 \quad \cdots \quad v_i^T \quad \cdots \quad 0 \,\middle|\, d_i^T \,\middle|\, 0 \,\middle|\, 0\right],\qquad 1\leqslant i\leqslant r,$$

where, for each $i$, $u_i, v_i$ are positive vectors of the same size and $c_i, d_i$ are nonnegative vectors not necessarily of the same size. Furthermore, $v_i^T u_i = (Py_i)^T(Px_i) = y_i^T x_i = 1$. Thus

$$
\begin{aligned}
PAP^T &= \sum_{i=1}^{r} (Px_i)(Py_i)^T \\
&= \sum_{i=1}^{r} \begin{bmatrix} 0 \\ \vdots \\ u_i \\ \vdots \\ 0 \\ \hline 0 \\ \hline c_i \\ \hline 0 \end{bmatrix} \left[\begin{array}{ccccc|c|c|c} 0 & \cdots & v_i^T & \cdots & 0 & d_i^T & 0 & 0 \end{array}\right] \\
&= \begin{bmatrix} u_1 v_1^T & & & u_1 d_1^T & & \\ & \ddots & & \vdots & 0 & 0 \\ & & u_r v_r^T & u_r d_r^T & & \\ & 0 & & 0 & 0 & 0 \\ c_1 v_1^T & \cdots & c_r v_r^T & \sum_{i=1}^{r} c_i d_i^T & 0 & 0 \\ & 0 & & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}
$$

where

$$
J = \begin{bmatrix} u_1 v_1^T & & \\ & \ddots & \\ & & u_r v_r^T \end{bmatrix}
$$

is a direct sum of positive idempotent matrices of rank one, as $u_i, v_i$ are positive vectors and $v_i^T u_i = 1$, and

$$C = \begin{bmatrix} c_1 v_1^T & \cdots & c_r v_r^T \end{bmatrix}$$

$$D^T = \begin{bmatrix} d_1 u_1^T & \cdots & d_r u_r^T \end{bmatrix}$$

are nonnegative matrices. ■

## 3. SUBGROUPS OF NONNEGATIVE MATRICES

We now consider subgroups of nonnegative matrices and related results from the operator-theoretic viewpoint. To put the situation in better perspective, we begin with results about subgroups of $\pi(K)$ for a proper cone $K$. It is known that the set $\{A \in \pi(K): AK = K\}$ is a group, called the *automorphism group* of $K$, and it is denoted by Aut $K$. (For reference, see Horne [25].)

THEOREM 3.1. *Let $P$ be a projection in $\pi(K)$. Then the maximal subgroup of $\pi(K)$ containing $P$ is isomorphic to the group* $\mathrm{Aut}(\mathrm{Im}\, P \cap K)$.

*Proof.* For each linear operator $X$ in $\mathrm{Aut}(\mathrm{Im}\, P \cap K)$, denote by $\Phi(X)$ the linear operator $XP$. Clearly $\Phi(X) \in \pi(K)$. Furthermore, for $X, Y \in \mathrm{Aut}(\mathrm{Im}\, P \cap K)$, $\Phi(XY) = XYP = XPYP$ [because $P$ is the identity on $\mathrm{Im}(YP) = \mathrm{Im}\, P$] $= \Phi(X)\Phi(Y)$. It is also obvious that $\Phi$ is one-to-one and $\Phi(\text{identity on } \mathrm{Im}\, P) = P$. Hence $\Phi(\mathrm{Aut}(\mathrm{Im}\, P \cap K))$ is a subgroup of $\pi(K)$ containing $P$ and is isomorphic to $\mathrm{Aut}(\mathrm{Im}\, P \cap K)$. Let $\mathcal{G}$ be any subgroup of $\pi(K)$ containing $P$, and let $A \in \mathcal{G}$. Then $\mathrm{Im}\, A = \mathrm{Im}\, P$, $A^{\#}$ exists and is in fact the inverse of $A$ in $\mathcal{G}$. Thus $A^{\#}A = P$, and so $A = AA^{\#}A = AP = A|_{\mathrm{Im}\, P} P$. By Theorem 2.2, $A|_{\mathrm{Im}\, P} \in \mathrm{Aut}(\mathrm{Im}\, P \cap K)$. Hence $A = \Phi(A|_{\mathrm{Im}\, P}) \in \Phi(\mathrm{Aut}(\mathrm{Im}\, P \cap K))$. Therefore $\mathcal{G} \subseteq \Phi(\mathrm{Aut}(\mathrm{Im}\, P \cap K))$. As $\mathcal{G}$ is an arbitrary subgroup containing $P$, $\Phi(\mathrm{Aut}(\mathrm{Im}\, P \cap K))$ is the maximal subgroup of $\pi(K)$ containing $P$. The proof is complete. ■

REMARK 3.2. By elementary semigroup theory (see, for instance, Clifford and Preston [15]), the maximal subgroup of $\pi(K)$ containing $P$, mentioned in Theorem 3.1, is in fact $\mathcal{H}_P$, the $\mathcal{H}$-class containing $P$. To keep our proof as elementary as possible, we have avoided the use of this fact.

From Theorem 3.1 it is clear that a maximal subgroup of $\pi(R^n_+)$ is isomorphic to the group $\mathrm{Aut}(\mathrm{Im}\, P \cap R^n_+)$, where $P$ is the identity element of the given maximal subgroup. Furthermore, we know $\mathrm{Im}\, P \cap R^n_+$ is an $r$-

dimensional simplicial cone, where $r=\operatorname{rank} P$. Just as in the case of the nonnegative orthant $R^r_+$, the automorphism group of $\operatorname{Im} P \cap R^n_+$ is isomorphic to the group of all $r \times r$ monomial matrices with positive nonzero entries. We have in fact established the following known result (Flor [19, Theorem 1], Plemmons [39, Corollary 1]).

COROLLARY 3.3. *Every maximal group of nonnegative matrices of rank r is isomorphic to the complete monomial group of degree r over the positive reals.*

The readers can observe that our above proof of Corollary 3.3 is essentially the same as that of Flor [19].

PROPOSITION 3.4. *Every bounded subgroup of $\pi(K)$ for a polyhedral cone K is isomorphic to a subgroup of some full symmetric group.*

*Proof.* Let $\mathcal{G}$ be a bounded subgroup of $\pi(K)$, where $K$ is a polyhedral cone. By Theorem 3.1, $\mathcal{G}$ is isomorphic to some subgroup of $\operatorname{Aut}(\operatorname{Im} P \cap K)$, where $P$ is the identity element of $\mathcal{G}$. So, without loss of generality, we assume that $\mathcal{G} \subseteq \operatorname{Aut} K$. Let $\Phi(x_1), \dots, \Phi(x_r)$ be all the extreme rays of $K$. If $A \in \mathcal{G}$, then $A$ induces a unique permutation $\pi_A$ on the set $\{\Phi(x_1), \dots, \Phi(x_r)\}$. Furthermore, the mapping $\pi: A \mapsto \pi_A$ is a group homomorphism. Let $A \in \operatorname{Ker} \pi$. Then, for each $i$, $1 \leqslant i \leqslant r$, $Ax_i = \lambda_i x_i$ for some positive scalar $\lambda_i$. Hence $A^t x_i = \lambda_i^t x_i$, $t=0, \pm 1, \pm 2, \dots$. Since the set $\{A^t : t \text{ is an integer}\}$ $(\subseteq \mathcal{G})$ is bounded, for each $i$ the set $\{\lambda_i^t : t \text{ is an integer}\}$ is also bounded. This forces $\lambda_i = 1$ for all $i$, and hence $A = I$. Therefore, the mapping $\pi$ is a group monomorphism, and $\mathcal{G}$ is isomorphic to a subgroup of the full symmetric group on $r$ letters. ∎

The above result is a minor improvement of Barker [1, Theorem 9], which in turn is an extension of Brown [11, Theorem 2], or Flor [19, Theorem 3]. Again the idea of our proof is essentially contained in Flor's paper. Flor [19] also gave a geometric proof of a known result about maximal subgroups of (row) stochastic matrices. However, it seems that it is easier to work with column stochastic matrices and then pass the result back to stochastic matrices by the transpose map. Recall that a nonnegative matrix is *row* (respectively, *column*) *stochastic* if all of its row (respectively, column) sums are equal to 1. As a linear operator, an $n \times n$ row stochastic matrix $A$ is characterized by $AR^n_+ \subseteq R^n_+$ and $Ae=e$, where $e$ is the vector with all entries 1; a column stochastic matrix $A$ is characterized by $A$ taking probability vectors to probability vectors. Clearly, $A$ is row stochastic iff $A^T$ is column stochastic.

PROPOSITION 3.5 (Flor [19], Schwarz [47], Wall [56]). *A maximal subgroup of column (or row) stochastic matrix of rank r is isomorphic to* $S(r)$, *the full symmetric group on r letters.*

*Proof.* Let $\mathcal{G}$ be a maximal subgroup of $n\times n$ column stochastic matrices of rank $r$, and let $E$ be the group identity. Then $\operatorname{Im} E\cap R^n_+$ is a simplicial cone (Proposition 2.5), say, generated by the extreme vectors $x_1,\dots,x_r$. These vectors may be chosen to be probability vectors. Let $A\in\mathcal{G}$. Then as a linear operator $A$ can be realized by, first, the nonnegative projection $E$, followed by an automorphism of the cone $\operatorname{Im} E\cap R^n_+$ (see proof of Theorem 3.1). Being column stochastic, $A$ sends probability vectors to probability vectors. So $A$ permutes the extreme vectors $x_1,\dots,x_r$ among themselves, and $A$ induces a permutation $\pi_A\in S(r)$. Clearly, for $A, B\in\mathcal{G}$, $\pi_{AB}=\pi_A\circ\pi_B$, and if $A\neq B$, then $\pi_A\neq\pi_B$. Conversely, if $\sigma\in S_r$, then $\sigma$ determines an automorphism $X$ of $\operatorname{Im} E\cap R^n_+$ which permutes the extreme vectors $x_1,\dots,x_r$ among themselves. Clearly $A=XE\in\pi(K)$; moreover $A$ sends probability vectors to probability vectors and hence is column stochastic. It is not difficult to see that $A$ also belongs to the maximal subgroup $\mathcal{G}$ and $\pi_A=\sigma$. Hence we have established an isomorphism between $\mathcal{G}$ and $S(r)$.

Next, let $\mathcal{H}$ be a maximal subgroup of row stochastic matrices of rank $r$. Then $\mathcal{H}^T=\{A^T\colon A\in\mathcal{H}\}$ is a maximal subgroup of column stochastic matrices of rank $r$. From the above, we know there exists a group isomorphism from $\mathcal{H}^T$ onto $S(r)$, say $\Phi$. Then it is straightforward to verify that the mapping $A\mapsto[\Phi(A^T)]^{-1}$ is an isomorphism from the group $\mathcal{H}$ onto the group $S(r)$. ∎

A similar proof of a known characterization of maximal subgroups of doubly stochastic matrices will be given in Section 7. We conclude this section with a known result concerning stochastic (or column stochastic) matrices with nonnegative Drazin pseudoinverses.

PROPOSITION 3.6 (Jain and Goel [27, Theorem 4]). *Let A be a column (or row) stochastic matrix with a nonnegative Drazin pseudoinverse. Then* $A^{(d)}=A^k$ *for some positive integer k and is column (or row) stochastic. The nilpotent-free part* $B=A^2A^{(d)}$ *is also column (or row) stochastic, and the nilpotent part N is such that the sum of entries in each column (or row) is zero.*

*Proof.* Let $A$ be a column stochastic matrix with $A^{(d)}\geqslant 0$, and let $k=$ index of $A$. Then $R^n=\operatorname{Im} A^k\oplus\operatorname{Ker} A^k$ and $A^{(d)}$ is given by $A^{(d)}|_{\operatorname{Ker} A^k}=0$ and $A^{(d)}|_{\operatorname{Im} A^k}=(A|_{\operatorname{Im} A^k})^{-1}$. By Theorem 2.3, $A$ maps the cone $\operatorname{Im} A^k\cap R^n_+$ onto itself. Since $\operatorname{Im} A^k$ is the image space of the nonnegative idempotent

$AA^{(d)}$, by Proposition 2.5 $\operatorname{Im} A^k \cap R^n_+$ is a simplicial cone, say generated by the extreme probability vectors $x_1, \dots, x_t$.

Now the restriction linear map $A|_{\operatorname{Im} A^k}$ permutes the extreme rays of $\operatorname{Im} A^k \cap R^n_+$; and as $A$ sends probability vectors to probability vectors, necessarily $A|_{\operatorname{Im} A^k}$ permutes the vectors $x_1, \dots, x_t$ among themselves. Let $v$ be the order of the corresponding permutation. Clearly $A^v|_{\operatorname{Im} A^k}$ is the identity map. Choose a positive integer $p$ such that $pv > k+1$. Then $A^{pv-1}|_{\operatorname{Im} A^k} = (A|_{\operatorname{Im} A^k})^{-1}$ and $A^{pv-1}|_{\operatorname{Ker} A^k} = 0$. Hence $A^{(d)} = A^u$, where $u = pv-1$ is a positive integer. The remaining assertions now follow readily. The row stochastic case also follows from the column stochastic case, as $(A^T)^{(d)} = (A^{(d)})^T$. ■

COROLLARY 3.7 (Jain, Goel, and Kwak [30, Corollary 4]). *If a column (or row) stochastic matrix $A$ has a nonnegative group inverse, then its group inverse must be a power of $A$ and hence is also column (or row) stochastic.*

## 4. NONNEGATIVE RANK FACTORIZATION

An $m \times n$ nonnegative matrix $A$ of rank $r$ is said to have a *nonnegative rank factorization* if there exist nonnegative matrices $B, C$ of order $m \times r$, $r \times n$ such that $A = BC$. (Then rank $B =$ rank $C = r$.) The concept has been useful in the study of nonnegative generalized inverses of nonnegative matrices (see Berman and Plemmons [7].) Thomas [54] and Campbell and Poole [14] have given equivalent conditions for a nonnegative matrix to have a nonnegative rank factorization. Below we give two equivalent conditions as well as a statement about "uniqueness" for nonnegative rank factorization. It will be seen that one of our conditions is essentially the same as that given in Campbell and Poole [14].

THEOREM 4.1. *Let $A$ be an $m \times n$ nonnegative matrix of rank $r$ ($\geq 1$). The following are equivalent:*

(i) *$A$ has a nonnegative rank factorization.*
(ii) *$A$ is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$ where $x_i \in R^m_+$ and $y_i \in R^n_+$.*
(iii) *There exists an ($r$-dimensional) simplicial cone $K$ such that $AR^n_+ \subseteq K \subseteq \operatorname{Im} A \cap R^m_+$.*

Furthermore, when the conditions are satisfied, the representation of $A$ given in (ii) is unique up to the order of its summands iff the simplicial cone K which satisfies (iii) is unique.

*Proof.* (i)$\Rightarrow$(ii): Let $A=BC$ be a nonnegative rank factorization of $A$. Write

$$B=[x_1 \quad \cdots \quad x_r] \quad \text{and} \quad C=\begin{bmatrix} y^T \\ \vdots \\ y_r^T \end{bmatrix}. \tag{$*$}$$

Then $A=x_1y_1^T+\cdots+x_ry_r^T$ with $x_i\in R_+^m$ and $y_i\in R_+^n$.

(ii)$\Rightarrow$(i): If $A=x_1y_1^T+\cdots+x_ry_r^T$ with $x_i\in R_+^m$ and $y_i\in R_+^n$, then $A=BC$, where $B,C$ are given by ($*$), is a nonnegative rank factorization of $A$.

(ii)$\Rightarrow$(iii): It is easily seen that the $r$-dimensional simplicial cone $K$ which is generated by the vectors $x_1,\ldots,x_r$ satisfies the requirement of condition (iii).

(iii)$\Rightarrow$(ii): Since both the cones $AR_+^n$ and $\operatorname{Im} A\cap R_+^m$ are $r$-dimensional, any simplicial cone $K$ which satisfies $AR_+^n\subseteq K\subseteq \operatorname{Im} A\cap R_+^m$ is necessarily $r$-dimensional. Let $K$ be one such cone, and let $x_1,\ldots,x_r$ be its extreme vectors. Since $AR_+^n\subseteq K$, by Lemma 2.7 there exist vectors $y_1,\ldots,y_r\in R_+^n$ such that $A=x_1y_1^T+\cdots+x_ry_r^T$.

*Last part*: If there are two different simplicial cones satisfying the requirement of condition (iii), then from the proof of (iii)$\Rightarrow$(ii) we obtain two genuinely different representations of $A$ as a sum of $r$ nonnegative rank-one matrices. Conversely, assume there exists a unique simplicial cone $K$, say, generated by $x_1,\ldots,x_r$, that satisfies the requirement of condition (iii). Let $A=u_1v_1^T+\cdots+u_rv_r^T$ with $u_i\in R_+^m$, $v_i\in R_+^n$. From the proof of (ii)$\Rightarrow$(iii), we know $u_1,\ldots,u_r$ generates a simplicial cone which satisfies condition (iii); so it is in fact the cone $K$. Hence, without altering the individual summands, $u_1v_1^T+\cdots+u_rv_r^T$ can be rewritten as $x_1y_1^T+\cdots+x_ry_r^T$ for some $y_i\in R_+^n$. But by Lemma 2.7, the vectors $y_i$ are uniquely determined. Hence the individual summands $x_iy_i^T$ (or $u_iv_i^T$) are uniquely determined. ■

It might be interesting to make the following two observations. First, from the proof of Theorem 4.1, it is clear that there is a close connection between a representation of $A$ as a sum of $r$ nonnegative rank-one matrices and a nonnegative rank factorization of $A$. From a given representation of $A$ as a sum of $r$ nonnegative rank-one matrices, we can construct in a natural way different nonnegative rank factorizations of $A$, which, however, are in some sense uniquely determined. For instance, if $A=x_1y_1^T+\cdots+x_ry_r^T$ with $x_i\in R_+^m$ and $y_i\in R_+^n$, then all the nonnegative rank factorizations of $A$ which arise from this representation of $A$ can be put into the form

$$\begin{bmatrix} \lambda_1 x_{\sigma(1)} & \lambda_2 x_{\sigma(2)} & \cdots & \lambda_r x_{\sigma(r)} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} y^T_{\sigma(1)} \\ \frac{1}{\lambda_2} y^T_{\sigma(2)} \\ \vdots \\ \frac{1}{\lambda_r} y^T_{\sigma(r)} \end{bmatrix},$$

where $\lambda_i > 0$ and $\sigma$ is a permutation in $S(r)$. This fact (together with Theorem 4.2) explains readily why if there is one nonnegative rank factorization $A = BC$ such that $B, C$ contain monomials of rank $r$, then all nonnegative rank factorizations of $A$ have this property (see Berman and Plemmons [7]). So, in some sense, condition (ii) of Theorem 4.1 is even simpler than the original definition of nonnegative rank factorization.

Second, the following is clearly another equivalent condition for $A$ to have a nonnegative rank factorization.

(iii′): There exists a simplicial cone $K$ such that $A^T R^m_+ \subseteq K \subseteq \operatorname{Im} A^T \cap R^n_+$.

However, it seems difficult to prove directly the equivalence of condition (iii′) with condition (iii). In contrast, condition (ii) of Theorem 4.1 is more "symmetrical," in the sense that its equivalence with its "dual" condition is obvious. This probably also "explains" why condition (ii) will be useful.

The following result is fundamental to our treatment of generalized inverses of nonnegative matrices.

THEOREM 4.2. *Let $A$ be an $m \times n$ nonnegative matrix of rank $r$ ($\geq 1$). If $A$ has a nonnegative (1)-inverse, then, except for the order of its summands, $A$ can be expressed uniquely as a sum of $r$ nonnegative rank-one matrices: $A = x_1 y_1^T + \cdots + x_r y_r^T$ with $x_i \in R^m_+$, $y_i \in R^n_+$; moreover, $x_1, \dots, x_r$ are extreme vectors of the $r$-dimensional simplicial cone* $\operatorname{Im} A \cap R^m_+$, *and $y_1, \dots, y_r$ are extreme vectors of the $r$-dimensional simplicial cone* $\operatorname{Im} A^T \cap R^n_+$.

*Proof.* From the discussion preceeding Proposition 2.8, $\operatorname{Im} A \cap R^m_+ = AR^n_+$ and is a simplicial cone. This is certainly the only simplicial cone that satisfies condition (iii) of Theorem 4.1. So the representation of $A$ as a sum of $r$ rank-one nonnegative matrices is "unique," say $A = x_1 y_1^T + \cdots x_r y_r^T$. Hence, from the proof of Theorem 4.1, necessarily $x_1, \dots, x_r$ are extreme vectors of

$\operatorname{Im} A \cap R_+^m$. Furthermore, $A^T = y_1 x_1^T + \cdots + y_r x_r^T$ is the "unique" representation of $A^T$ as a sum of $r$ nonnegative rank-one matrices, and $A^T$ has a nonnegative (1)-inverse. So likewise, $y_1, \ldots, y_2$ are extreme vectors of $\operatorname{Im} A^T \cap R_+^n$. ■

There has been some interest in seeking a general procedure to determine the existence of a nonnegative rank factorization and an algorithm to find the factorization if it exists. Obviously the problem of finding a nonnegative rank factorization for an $m \times n$ nonnegative matrix $A$ of rank $r$ is equivalent to seeking the nonnegative solutions of a system of $mn$ equations of second degree with $(m+n)\times r$ unknowns. As such, the general problem seems intractable. Theorem 4.1 and its proof direct our attention to the geometrical problem of finding a simplicial cone $K$ that satisfies $AR_+^n \subseteq K \subseteq \operatorname{Im} A \cap R_+^m$. This approach is helpful in special cases, and in constructing examples or counterexamples of nonnegative rank factorizations. But it does not help to solve the general problem. (We have tried to translate the geometrical problem into an algebraic problem, assuming that we already know the extreme vectors of the cones $AR_+^n$ and $\operatorname{Im} A \cap R_+^m$. Even then we end up with the problem of finding the nonnegative solutions of the same type of equations. So we are not doing any better.) Recently, Campbell and Poole [14] gave an effective algorithm to compute a nonnegative rank factorization of $A$ whenever $A$ has a nonnegative (1)-inverse. The reader may find it interesting to understand the geometrical meaning of their algorithm. Actually their algorithm finally leads to a cone $C$ with extreme vectors, say $y_1, \ldots, y_t$, that satisfies the following: $A^T R_+^m \subseteq C \subseteq \operatorname{Im} A^T \cap R_+^n$, and for $i \neq j$, $\Phi(y_i) \not\subseteq \Phi(y_j)$, where $\Phi(y_i)$ denotes the face of $R_+^n$ generated by $y_i$. (Their algorithm works with row vectors of $A$.) Clearly, if the resulting cone $C$ is simplicial (i.e. $t = r$), then by Theorem 4.1 $A$ has a nonnegative rank factorization. This certainly covers the most interesting case when $A$ has a nonnegative (1)-inverse. However, then we might dispense with their algorithm. For in this case $A$ contains an $r \times r$ monomial submatrix (see Corollary 5.3), and the corresponding columns of $A$ are just the extreme vectors of the simplicial cone $AR_+^n$; hence, it is easy to find a nonnegative rank factorization of $A$ (recall the proof of Lemma 2.7).

EXAMPLE 4.3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

It can be shown that $\operatorname{Im} A \cap R^4_+$ is the polyhedral cone with extreme vectors $x_1 = (1,0,1,0)^T$, $x_2 = (0,1,0,1)^T$, $x_3 = (1,0,0,1)^T$, and $x_4 = (0,1,1,0)^T$; $AR^4_+$ is the simplicial cone generated by the vectors $x_1, x_2, x_3$, and is in fact the only simplicial cone $K$ that satisfies $AR^4_+ \subseteq K \subseteq \operatorname{Im} A \cap R^4_+$. (The reader can easily verify the last assertion with the help of a picture.) So by Theorem 4.1, $A$ can be expressed as a sum of three nonnegative rank-one matrices in essentially one way, namely,

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} [1 \quad 0 \quad 0 \quad 1] + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} [0 \quad 1 \quad 0 \quad 1] + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} [0 \quad 0 \quad 1 \quad 0].$$

Also

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is a nonnegative rank factorization of $A$. However, $A$ has no nonnegative (1)-inverse, as $AR^4_+ \neq \operatorname{Im} A \cap R^4_+$. Hence nonnegative matrices with nonnegative (1)-inverse are *not* characterized by the property of being expressible "uniquely" as a sum of $r$ nonnegative rank-one matrices, where $r$ is the rank of the given matrix (cf. Theorem 4.2).

## 5. NONNEGATIVE GENERALIZED INVERSES

We now consider the problem of characterizing nonnegative matrices with various kinds of nonnegative generalized inverses. For a nonnegative matrix $A$ of rank $r$ with a nonnegative (1)-inverse, we shall distinguish the various kinds of nonnegative generalized inverses $A$ might have by characterizing the corresponding families of vectors $x_1, \dots, x_r$ and $y_1, \dots, y_r$ which appear in the "unique" representation of $A$ as a sum of $r$ nonnegative rank-one matrices (see Theorem 4.2). Various kinds of characterization of $A$ in matrix block forms will then follow readily.

LEMMA 5.1. *Let $x_1, \dots, x_r \in R^n_+$. The following are equivalent:*

*(i) The vectors $x_1, \dots, x_r$ are extreme vectors of the $r$-dimensional simplicial cone* $\operatorname{span}\{x_1, \dots, x_r\} \cap R^n_+$,

(ii) *Denote by* $x_{ij}$ *the* $j$th *component of the vector* $x_i$. *For each* $i$, *there exists* $k_i$ *such that* $x_{ik_i} \neq 0$ *but* $x_{jk_i} = 0$ *for* $j \neq i$.

(iii) *There exists a permutation matrix P such that the vectors* $Px_1, \dots, Px_r$ *are partitioned conformally*:

$$Px_1 = \begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \\ v_1 \end{bmatrix}, \dots, \qquad Px_i = \begin{bmatrix} 0 \\ \vdots \\ u_i \\ \vdots \\ 0 \\ v_i \end{bmatrix}, \dots, \qquad Px_r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_r \\ v_r \end{bmatrix}$$

*where* $u_1, \dots, u_r$ *are positive vectors, possibly of different size, and* $v_1, \dots, v_r$ *are nonnegative vectors and may not appear.*

(iv) *There exist vectors* $y_1, \dots, y_r \in R^n_+$ *such that* $x_i^T y_j = \delta_{ij}$.

*Proof.* (ii)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv) are fairly obvious.

(iv)$\Rightarrow$(i): Suppose that $y_1, \dots, y_r$ are vectors in $R^n_+$ satisfying $x_i^T y_j = \delta_{ij}$. Then $P = x_1 y_1^T + \cdots + x_r y_r^T$ is a nonnegative idempotent matrix of rank $r$ with image space span$\{x_1, \dots, x_r\}$, and by Theorem 4.2, $x_1, \dots, x_r$ are the extreme vectors of the $r$-dimensional simplicial cone $\operatorname{Im} P \cap R^n_+ = \operatorname{span}\{x_1, \dots, x_r\} \cap R^n_+$.

(i)$\Rightarrow$(iv): By Proposition 2.5 there exists a nonnegative idempotent matrix $P$ with image space span$\{x_1, \dots, x_r\}$. By Proposition 2.6 and Theorem 4.2, $P$ is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$ where $y_i \in R^n_+$ satisfies $x_i^T y_j = \delta_{ij}$. ■

THEOREM 5.2. *Let A be an* $m \times n$ *nonnegative matrix of rank* $r$. *The following are equivalent*:

(i) *A has a nonnegative* (1)*-inverse.*

(ii) *A is expressible as* $x_1 y_1^T + \cdots + x_r y_r^T$ *where* $x_1, \dots, x_r$ *are vectors in* $R^m_+$ *satisfying the equivalent conditions of Lemma* 5.1, *and* $y_1, \dots, y_r$ *are vectors in* $R^n_+$ *satisfying similar conditions.* (*Then a nonnegative matrix is a semiinverse of A iff it can be written as* $a_1 b_1^T + \cdots + a_r b_r^T$ *where* $a_i \in R^n_+$, $b_i \in R^m_+$, $x_i^T b_j = \delta_{ij}$, *and* $y_i^T a_j = \delta_{ij}$.)

(iii) *There exist an* $m \times m$ *permutation matrix P and an* $n \times n$ *permutation matrix Q such that the matrix* $PAQ^T$ *can be partitioned in the following block* *form*:

$$\begin{bmatrix} u_1c_1^T & 0 & \cdots & 0 & u_1d_1^T \\ 0 & u_2c_2^T & \cdots & 0 & u_2d_2^T \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_rc_r^T & u_rd_r^T \\ v_1c_1^T & v_2c_2^T & \cdots & v_rc_r^T & N \end{bmatrix},$$

*where* $u_1, \dots, u_r, c_1, \dots, c_r$ *are positive vectors, possibly of different size,* $v_1, \dots, v_r$ *are nonnegative vectors of the same size,* $d_1, \dots, d_r$ *are nonnegative vectors also of the same size, and* $N$ *is a nonnegative matrix. (Then* $N = \sum_{i=1}^r v_i d_i^T$.*) It is understood that the blocks* $v_ic_i^T$, $u_id_i^T$, *and* $N$ *may not appear.*

*Proof.* (i)$\Rightarrow$(ii): By Theorem 4.2, $A = x_1y_1^T + \cdots + x_ry_r^T$, where $x_1, \dots, x_r$ are extreme vectors of the simplicial cone $\operatorname{Im} A \cap R_+^m$ and $y_1, \dots, y_n$ are extreme vectors of the simplicial cone $\operatorname{Im} A^T \cap R_+^n$. Clearly, the vectors $x_1, \dots, x_r$ (as well as $y_1, \dots, y_r$) satisfy the equivalent conditions of Lemma 5.1.

Let $X$ be a nonnegative semiinverse of $A$. Then $X$ has a nonnegative (1)-inverse, namely $A$. So by Theorem 4.2, $X$ is expressible as $a_1b_1^T + \cdots + a_rb_r^T$ where $a_1, \dots, a_r$ (respectively $b_1, \dots, b_r$) are extreme vectors of $\operatorname{Im} X \cap R_+^n$ (respectively $\operatorname{Im} X^T \cap R_+^m$). From the geometrical properties of semiinverses (see, for instance, Tam [52, Fact 2.1]), we see that $X$ takes $\operatorname{Im} A \cap R_+^m$ one-to-one onto $\operatorname{Im} X \cap R_+^n$, and hence $A$ takes $\operatorname{Im} X \cap R_+^n$ one-to-one $\operatorname{Im} A \cap R_+^m$. Thus $A$ sends extreme vectors of $\operatorname{Im} X \cap R_+^n$ to extreme vectors of $\operatorname{Im} A \cap R_+^m$. Without loss of generality, assume $x_i = Aa_i$ for each $i$. Then, for each $i$, $(x_1y_1^T + \cdots + x_ry_r^T)a_i = x_i$; hence $y_i^Ta_j = \delta_{ij}$ for all $i, j$. Moreover, since $AX = (x_1y_1^T + \cdots + x_ry_r^T)(a_1b_1^T + \cdots + a_rb_r^T) = x_1b_1^T + \cdots + x_rb_r^T$ is a nonnegative idempotent matrix, as $X$ is a nonnegative (1)-inverse of $A$, by Proposition 2.6 we also have $x_i^Tb_j = \delta_{ij}$.

(ii)$\Rightarrow$(i): By condition (iv) of Lemma 5.1, there exist vectors $b_1, \dots, b_r \in R_+^m$ such that $x_i^Tb_j = \delta_{ij}$ and vectors $a_1, \dots, a_r \in R_+^n$ such that $y_i^Ta_j = \delta_{ij}$. Let $X = a_1b_1^T + \cdots + a_rb_r^T$. Then

$$\begin{aligned} AXA &= (x_1y_1^T + \cdots + x_ry_r^T)(a_1b_1^T + \cdots + a_rb_r^T)(x_1y_1^T + \cdots + x_ry_r^T) \\ &= (x_1b_1^T + \cdots + x_rb_r^T)(x_1y_1^T + \cdots + x_ry_r^T) \\ &= x_1y_1^T + \cdots + x_ry_r^T \\ &= A. \end{aligned}$$

Similarly, $XAX=X$. Hence $X$ is a nonnegative semiinverse of $A$.

(ii)$\Rightarrow$(iii): By condition (iii) of Lemma 5.1, for the family of vectors $x_1,\dots,x_r$, there exists an $m\times m$ permutation matrix $P$ such that

$$(Px_i)^T=\begin{bmatrix}0 & \cdots & u_i^T & \cdots & 0 & v_i^T\end{bmatrix}, \qquad 1\leqslant i\leqslant r,$$

where the vectors $u_i, v_i$ satisfy the requirements there. Similarly, there exists an $n\times n$ permutation matrix $Q$ such that

$$(Qy_i)^T=\begin{bmatrix}0 & \cdots & c_i^T & \cdots & 0 & d_i^T\end{bmatrix}, \qquad 1\leqslant i\leqslant r,$$

where the vectors $c_i, d_i$ satisfy similar requirements. Thus

$$\begin{aligned}PAQ^T&=P(x_1y_1^T+\cdots+x_ry_r^T)Q^T\\&=(Px_1)(Qy_1)^T+\cdots+(Px_r)(Qy_r)^T\\&=\sum_{i=1}^{r}\begin{bmatrix}0\\ \vdots\\ u_i\\ \vdots\\ 0\\ v_i\end{bmatrix}\begin{bmatrix}0 & \cdots & c_i^T & \cdots & 0 & d_i^T\end{bmatrix}\end{aligned} \qquad (*)$$

and can be put into the required block form.

(iii)$\Rightarrow$(ii): Suppose there exist permutation matrices $P$ and $Q$ such that $PAQ^T$ is of the required form. Since the vectors $[u_1^T\ 0\ \cdots\ 0\ v_1^T]^T,\dots,[0\ \cdots\ u_r^T\ v_r^T]^T$ generate the column space of $PAQ^T$, it is not difficult to see that $N=\Sigma v_i d_i^T$. Hence $PAQ^T$ can be rewritten as in $(*)$. Thus $A=\Sigma_{i=1}^r x_i y_i^T$, where

$$x_i^T=\begin{bmatrix}0 & \cdots & u_i^T & \cdots & 0 & v_i^T\end{bmatrix}P,$$

$$y_i^T=\begin{bmatrix}0 & \cdots & c_i^T & \cdots & 0 & d_i^T\end{bmatrix}Q.$$

Clearly the vectors $x_1,\dots,x_r$ (as well as the vectors $y_1,\dots,y_r$) satisfy condition (iii) of Lemma 5.1. ∎

It may be interesting to compare our characterization of a nonnegative matrix having a nonnegative (1)-inverse in matrix block form with that of Jain and Snyder [31, Theorem 1]. As can be seen, the block form given there looks like that for a canonical nonnegative idempotent matrix as given by Flor (see Theorem 2.9). However, in this case it seems somehow unsatisfactory, for the following reasons. First, the matrix $PAQ^T$ is usually not square. The zero blocks which appear in the block form do not have any natural sizes. Second, the matrix $J$ (given there) can be assumed simply to be a direct sum of matrices of type (I) only. (See the definitions for matrices of types (I) and (II) given in [31].) The point is, after permuting rows (or columns), a matrix of type (II) can be expressed as a direct sum of matrices of type (I).

Actually, if we let

$$J=\begin{bmatrix} u_1c_1^T & & & \\ & u_2c_2^T & & \\ & & \ddots & \\ & & & u_rc_r^T \end{bmatrix},$$

then it is not difficult to show that our matrix block form given in condition (iii) of Theorem 5.2 can be rewritten as

$$\begin{bmatrix} J & JD \\ CJ & CJD \end{bmatrix},$$

where $C$, $D$ are nonnegative matrices.

COROLLARY 5.3 (Jain and Snyder [31, Corollary 4]). *For an $m\times n$ nonnegative matrix $A$ of rank $r$, the following are equivalent*:

(i) *$A$ has a nonnegative (1)-inverse,*

(ii) *$A$ has a (1)-inverse of the form $D_1A^TD_2$, where $D_1$, $D_2$ are nonnegative diagonal matrices of rank $r$,*

(iii) *$A$ has a monomial submatrix of rank $r$,*

(iv) *$A$ has a nonnegative rank factorization $FG$ when $F, G$ have monomial submatrices of rank $r$. (Moreover, every nonnegative rank factorization of $A$ has this form.)*

*Proof.* (i)$\Rightarrow$(ii): By condition (ii) of Theorem 5.2 $A$ can be expressed as $x_1y_1^T+\cdots+x_ry_r^T$ where $x_1,\dots,x_r$ (respectively $y_1,\dots,y_r$) are nonnegative vectors satisfying the equivalent conditions of Lemma 5.1. In view of condition (ii) of Lemma 5.1, there exist nonnegative vectors $b_1,\dots,b_r$ (respectively $a_1,\dots,a_r$), each of which has exactly one positive entry, satisfying $x_i^T b_j = \delta_{ij}$ (respectively $y_i^T a_j = \delta_{ij}$). From condition (ii) of Theorem 5.2, the nonnegative matrix $a_1 b_1^T + \cdots + a_r b_r^T$ obtained in this way is a semiinverse of $A$. Furthermore,

$$\begin{aligned} a_1 b_1^T &+ \cdots + a_r b_r^T \\ &= (a_1 a_1^T + \cdots + a_r a_r^T)(y_1 x_1^T + \cdots + y_r x_r^T)(b_1 b_1^T + \cdots + b_r b_r^T) \\ &= D_1 A^T D_2, \end{aligned}$$

where $D_1 = a_1 a_1^T + \cdots + a_r a_r^T$ and $D_2 = b_1 b_1^T + \cdots + b_r b_r^T$ are nonnegative diagonal matrices of rank $r$.

(ii)$\Rightarrow$(i): Obvious.

(i)$\Rightarrow$(iii): By condition (iii) of Theorem 5.2.

(iii)$\Rightarrow$(i): Then there exist permutation matrices $P$ and $Q$ such that $PAQ$ is of the form

$$\begin{bmatrix} M & * \\ * & * \end{bmatrix}$$

where $M$ is an $r \times r$ monomial matrix. It is obvious that the above matrix is in the matrix block form of Theorem 5.2, condition (iii). So $A$ has a nonnegative (1)-inverse.

(i)$\Rightarrow$(iv): Refer to the discussion after Theorem 4.1. ■

THEOREM 5.4. *Let $A$ be an $n \times n$ nonnegative matrix of rank $r$. The following are equivalent*:

(i) *The group inverse $A^\#$ of $A$ exists and is nonnegative.*

(ii) *$A$ is expressible as $\lambda_1 x_1 y_1^T + \cdots + \lambda_r x_r y_r^T$, where $\lambda_i > 0$, $x_i, y_i \in R^n_+$, and for some permutation $\sigma \in S(r)$, $y^T_{\sigma(i)} x_i = \delta_{ij}$. Then*

$$A^\# = \sum_{i=1}^{r} \frac{1}{\lambda_{\sigma(i)}} x_i y^T_{\sigma^2(i)}.$$

(iii) *There exists an $n\times n$ permutation matrix $P$ such that*

$$PAP^T=\begin{bmatrix} A_1 & 0 & \cdots & 0 & U_1 \\ 0 & A_2 & \cdots & 0 & U_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_s & U_s \\ V_1 & V_2 & \cdots & V_s & N \end{bmatrix} \tag{\#}$$

*and the following conditions are satisfied:*

(a) $N=\sum_{i=1}^{s} N_i$.

(b) *Each matrix* $\begin{bmatrix} A_i & U_i \\ V_i & N_i \end{bmatrix}$ *satisfies one of the following two statements:*

(I) $A_i=u_iv_i^T$, *where* $u_i, v_i$ *are positive vectors of the same size;* $U_i=u_ic_i^T$, $V_i=d_iv_i^T$, *and* $N_i=d_ic_i^T$, *where* $c_i, d_i$ *are nonnegative vectors of the same size.*

(II) $A_i$ *has the form*

$$A_i=\begin{bmatrix} 0 & u_{i1}v_{i2}^T & 0 & \cdots & 0 \\ 0 & 0 & u_{i2}v_{i2}^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & u_{in_i-1}v_{in_i}^T \\ u_{in_i}v_{i1}^T & 0 & 0 & \cdots & 0 \end{bmatrix},$$

*where the zero blocks on the main diagonals are square and* $u_{ij}, v_{ij}$ *are positive vectors;*

$$U_i=\begin{bmatrix} u_{i1}c_{i1}^T \\ u_{i2}c_{i2}^T \\ \vdots \\ u_{in_i}c_{in_i}^T \end{bmatrix}$$

*for some nonnegative vectors* $c_{ij}$;

$$V_i=\begin{bmatrix} d_{i1}v_{i1}^T & d_{i2}v_{i2}^T & \cdots & d_{in_i}v_{in_i}^T \end{bmatrix}$$

*for some nonnegative vectors* $d_{ij}$; *and*

$$N_i = \sum_{j=1}^{n_i} d_{ij} c_{i,j-1}^T.$$

(iv) *$A$ has a nonnegative rank factorization $A = BG$, where $GB$ is monomial.* (*Then* $A^{\#} = B(GB)^{-2}G$.)

*Proof.* (i)$\Rightarrow$(ii): By Theorem 4.2 $A$ is expressible as $x_1 z_1^T + \cdots + x_r z_r^T$ where $x_1, \ldots, x_r$ are extreme vectors of the $r$-dimensional simplicial cone $\operatorname{Im} A \cap R_+^n$ and $z_1, \ldots, z_r \in R_+^n$. By Theorem 2.2, $A|_{\operatorname{Im} A}$ belongs to the automorphism group of $\operatorname{Im} A \cap R_+^n$. So for some permutation $\sigma$ and some positive scalars $\lambda_i$, $Ax_i = \lambda_{\sigma(i)} x_{\sigma(i)}$. Hence $z_{\sigma(i)}^T x_i = \lambda_{\sigma(i)}$ and $z_{\sigma(i)}^T x_j = 0$ for $j \neq i$. Let $y_i = (1/\lambda_i) z_i$. Then $A = \sum_{i=1}^r \lambda_i x_i y_i^T$ with $y_{\sigma(j)}^T x_i = \delta_{ij}$ as required.

(ii)$\Rightarrow$(i): It can be verified directly that the nonnegative matrix

$$\sum_{i=1}^{r} \frac{1}{\lambda_{\sigma(i)}} x_i y_{\sigma^2(i)}^T$$

satisfies the conditions for being the group inverse of $A$.

(ii)$\Rightarrow$(iii): Obviously (ii) can be stated equivalently as: $A$ is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$, where $x_i, y_i \in R_+^n$, and for some permutation $\sigma \in S(r)$,

$$y_{\sigma(j)}^T x_i \text{ is } \begin{cases} \text{nonzero} & \text{if } \; i = j, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For each $i$, $1 \leq i \leq r$, let

$$M(i) = \{k : x_{ik} \neq 0 \text{ and } y_{\sigma(i)k} \neq 0\},$$

where $x_{ik}$ denotes the $k$th component of $x_i$, and $y_{ik}$ has similar meaning. Consider the principal submatrix of $A$ whose rows and columns are determined by the set $\bigcup_{i=1}^r M(i)$. Permute the rows and columns of this matrix simultaneously so that rows (hence also columns) which are determined by the same $M(i)$ are grouped together. In view of (1), it is not difficult to see that the resulting matrix is in "block-monomial" form, where each nonzero block is a positive square matrix of rank one. But every permutation can be written as a product of disjoint cycles, so after permuting simultaneously the rows and columns suitably, this resulting matrix can be brought into the form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_s \end{bmatrix},$$

where $A_i$ are square blocks satisfying either (I) or (II) of condition (b). Now we can find a permutation matrix $P$ such that $PAP^T$ is of the form as given in (#) of condition (iii).

As $A$ is a sum of the rank-one matrices $x_i y_i^T$, it is easily seen that each of the matrices $U_i, V_i$ satisfies the requirement of the theorem. That $N = \sum_{i=1}^s N_i$ is also obvious. We omit the details.

(iii)$\Rightarrow$(ii): Fairly obvious.

(ii)$\Rightarrow$(iv): As in the proof of (ii)$\Rightarrow$(iii), $A$ is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$, where $x_i, y_i \in R^n_+$ and for some permutation $\sigma \in S(r)$, (1) is satisfied. Let

$$B = [x_1 \quad x_2 \quad \cdots \quad x_r], \qquad G = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_r^T \end{bmatrix}. \tag{2}$$

Then $BG$ is a nonnegative rank factorization of $A$. Moreover,

$$GB = \begin{bmatrix} y_1^T \\ \vdots \\ y_r^T \end{bmatrix} [x_1 \quad \cdots \quad x_r] = \begin{bmatrix} y_1^T x_1 & \cdots & y_1^T x_r \\ \vdots & & \vdots \\ y_r^T x_1 & \cdots & y_r^T x_r \end{bmatrix}, \tag{3}$$

and, in view of (1), $GB$ is monomial.

(iv)$\Rightarrow$(ii): Let $BG$ be a nonnegative rank factorization of $A$ such that $GB$ is monomial. Rewrite $B$ and $G$ as in (2). Then $A = x_1 y_1^T + \cdots + x_r y_r^T$. Furthermore, since $GB$ is monomial, by (3) there exists a permutation $\sigma \in S(r)$ such that (1) holds. ∎

REMARK 5.5. Condition (iv) of Theorem 5.4 has appeared in Berman and Plemmans [8]. Condition (iii) is similar to the condition in Theorem 2.5 of Haynsworth and Wall [23] and contains it as a special case. The characterization in matrix block form given in Jain and Snyder [31, Corollary 1] can also be derived from condition (iii) [or (ii)].

LEMMA 5.6. *Let* $x_1,\dots,x_r \in R^n_+$. *The following are equivalent*:

(i) $x_1,\dots,x_r$ *are mutually orthogonal nonzero vectors.*

(ii) *Denote by* $x_{ij}$ *the* $j$*th component of the vector* $x_i$. *For each* $i$, *there exists* $k_i$ *such that* $x_{ik_i} \neq 0$ *but* $x_{jk_i}=0$ *for* $j\neq i$. *Furthermore, no two of the vectors* $x_1,\dots,x_r$ *have nonzero entries at the same position.*

(iii) *There exists an* $n\times n$ *permutation matrix* $P$ *such that the vectors* $Px_1,\dots,Px_r$ *are partitioned conformally*:

$$Px_1=\begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\dots,\qquad Px_i=\begin{bmatrix} 0 \\ \vdots \\ u_i \\ \vdots \\ 0 \\ 0 \end{bmatrix},\dots,\qquad Px_r=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_r \\ 0 \end{bmatrix},$$

*where* $u_1,\dots,u_r$ *are positive vectors, possibly of different size, and the last zero blocks may not appear.*

(iv) $\operatorname{span}\{x_1,\dots,x_r\}\cap R^n_+$ *is an* $r$*-dimensional simplicial, self-dual cone with* $x_1,\dots,x_r$ *as extreme vectors.*

*Proof.* The equivalences of (i), (ii), and (iii) are obvious. The implication (iv)$\Rightarrow$(i) is also obvious.

(i),(ii)$\Rightarrow$(iv): Since (ii) is satisfied, by Lemma 5.1, $\operatorname{span}\{x_1,\dots,x_r\}\cap R^n_+$ is a simplicial cone with extreme vectors $x_1,\dots,x_r$. As the vectors $x_1,\dots,x_r$ are mutually orthogonal, the cone $\operatorname{span}\{x_1,\dots,x_r\}\cap R^n_+$ is clearly also self-dual. ■

A square matrix $A=[a_{ij}]$ is called *0-symmetric* if $a_{ij}=0$ implies $a_{ji}=0$. $A$ is called *weakly 0-symmetric* if the $i$th row of $A$ is zero iff the $i$th column of $A$ is zero. A nonnegative idempotent matrix $E$ is weakly 0-symmetric iff it is 0-symmetric. The reason is, if $E$ is weakly 0-symmetric, then by Theorem 2.9 (Flor's characterization of nonnegative idempotent matrices), $E$ is permutationally similar to a matrix of the form

$$\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$$

where $J$ is a direct sum of positive rank-one idempotent matrices. Thus $E$ is also 0-symmetric.

PROPOSITION 5.7. *Let $H$ be an $r$-dimensional subspace of $R^n$. The following are equivalent*:

(i) *$H \cap R^n_+$ is an $r$-dimensional simplicial self-dual cone*,

(ii) *The orthogonal projection of $R^n$ onto $H$ is nonnegative* (*with respect to* $R^n_+$),

(iii) *There exists a column stochastic idempotent with image space $H$*,

(iv) *There exists a nonnegative 0-symmetric idempotent with image space $H$*,

(v) *There exists a nonnegative weakly 0-symmetric idempotent with image space $H$.*

*Proof.* The equivalence of (iv) and (v) follows from the remark preceding the proposition.

(i)$\Rightarrow$(ii): Assume that $x_1, \dots, x_r$ are the extreme vectors of $H \cap R^n_+$ such that $x_i^T x_j = \delta_{ij}$. Then the nonnegative matrix $x_1 x_1^T + \cdots + x_r x_r^T$ is the orthogonal projection of $R^n$ on $H$.

(i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv): Assume that $x_1, \dots, x_r$ are the extreme probability vectors of $H \cap R^n_+$. Then these vectors satisfy condition (iii) of Lemma 5.6. Let $y_1, \dots, y_r \in R^n_+$ be given by

$$(Py_i)^T = \left[\begin{array}{c|c|ccc|c|c|ccc} 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 & \frac{1}{r} & \cdots & \frac{1}{r} \end{array}\right], \qquad 1 \leqslant i \leqslant r,$$

where the vectors $Py_1, \dots, Py_r$ are partitioned conformally and in the same way as the vectors $Px_1, \dots, Px_r$. Then the nonnegative matrix $x_1 y_1^T + \cdots + x_r y_r^T$ is a column stochastic idempotent with image space $H$.

Let $u_1, \dots, u_r \in R^n_+$ be given by

$$(Pu_i)^T = [\,0 \mid \cdots \mid 1 \quad \cdots \quad 1 \mid \cdots \mid 0\,], \qquad 1 \leqslant i \leqslant r,$$

where the vectors $Pu_1, \dots, Pu_r$ are partitioned conformally and in the same way as the vectors $Px_1, \dots, Px_r$. Then $x_1 u_1^T + \cdots + x_r u_r^T$ is a nonnegative 0-symmetric idempotent with image space $H$.

(ii)$\Rightarrow$(i): Let $Q$ denote the orthogonal projection of $R^n$ on $H$. By Proposition 2.6, $Q = x_1 y_1^T + \cdots + x_r y_r^T$ for some $x_i, y_i \in R^n_+$ such that $y_i^T x_j = \delta_{ij}$. As is well known, a projection is orthogonal iff it is symmetric. Thus $Q=Q^T$, i.e.

$$x_1y_1^T+\cdots+x_ry_r^T=y_1x_1^T+\cdots+y_rx_r^T.$$

Since $Q$ has a nonnegative (1)-inverse (namely, itself), the representation of $Q$ as a sum of $r$ nonnegative rank-one matrices is unique, except for the order of its summands. Hence each $x_iy_i^T$ is equal to some $y_jx_j^T$. But for $i\neq j$, $x_i$ is orthogonal to $y_j$. This forces $x_iy_i^T=y_ix_i^T$ for each $i$. Hence each $y_i$ is a positive multiple of $x_i$. So it is clear that the cone $H\cap R^n_+$, which is generated by $x_1,\dots,x_r$, is a simplicial, self-dual cone.

$\sim$(i)$\Rightarrow\sim$(iii),$\sim$(iv): In view of Proposition 2.5, we may assume that $H\cap R^n_+$ is an $r$-dimensional simplicial cone, which is not self-dual and is generated by, say, the extreme vectors $x_1,\dots,x_r$. By Lemmas 5.1 and 5.6, two of the vectors $x_1,\dots,x_r$ have nonzero entries at the same position. Without loss of generality, assume that $x_{1k}$ and $x_{2k}$ are both nonzero. Let $P$ be any nonnegative idempotent with image space $H$. Then $P=x_1y_1^T+\cdots+x_ry_r^T$ for some $y_1,\dots,y_r\in R^n_+$ such that $y_i^Tx_j=\delta_{ij}$. It is not difficult to see that the conditions $y_i^Tx_j=\delta_{ij}$ imply that each vector $y_i$ has zero entry at the $k$th position. Hence the $k$th column of $P$ is zero, and $P$ cannot be column stochastic. Observe however that the $k$th row of $P$ is nonzero, and hence $P$ is also not weakly 0-symmetric. ■

COROLLARY 5.8. *An $n\times n$ nonnegative matrix $A$ of rank $r$ is idempotent and symmetric iff it is expressible as $x_1x_1^T+\cdots+x_rx_r^T$ where $x_i\in R^n_+$ satisfying $x_i^Tx_j=\delta_{ij}$.*

*Proof.* The "if" part is trivial. The "only if" part follows from the proof of (ii)$\Rightarrow$(i) of Proposition 5.7. ■

THEOREM 5.9. *Let $A$ be an $m\times n$ nonnegative matrix of rank $r$. The following are equivalent:*

(i) *$A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$, where $x_1,\dots,x_r\in R^m_+$ satisfy the equivalent conditions of Lemma* 5.6 *and $y_1,\dots,y_r\in R^n_+$ satisfy similar conditions.*

(ii) *There exist an $m\times m$ permutation matrix $P$ and an $n\times n$ permutation matrix $Q$ such that*

$$PAQ^T=\begin{bmatrix} u_1v_1^T & & & 0\\ & \ddots & & \vdots\\ & & u_rv_r^T & 0\\ & 0 & & 0\end{bmatrix},$$

*where $u_i, v_i$ are positive vectors, possibly of different size. It is understood that the zero blocks may not exist.*

(iii) *A has a nonnegative Moore-Penrose inverse.*

(iv) *There exists a nonnegative semiinverse X of A such that AX and XA are both 0-symmetric.*

*Proof.* (i)$\Leftrightarrow$(ii): Similar to the proof of (ii)$\Leftrightarrow$(iii) in Theorem 5.2; here we need condition (iii) of Lemma 5.6.

(i)$\Rightarrow$(iii): It can be verified directly that the nonnegative matrix

$$\frac{y_1x_1^T}{\|x_1\|^2\|y_1\|^2}+\cdots+\frac{y_rx_r^T}{\|x_r\|^2\|y_r\|^2}$$

is the Moore-Penrose inverse of $A$.

(iii)$\Rightarrow$(iv): Obvious.

(iv)$\Rightarrow$(i): Since $A$ has a nonnegative (1)-inverse, $A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$ where $x_1,\dots,x_r$ (respectively $y_1,\dots,y_r$) are the extreme vectors of the simplicial cone $\operatorname{Im} A\cap R^m_+$ (respectively $\operatorname{Im} A^T\cap R^n_+$). Now the image space of $A$ and the image space of the 0-symmetric idempotent $AX$ are the same. Thus $\operatorname{Im} A\cap R^m_+=\operatorname{Im}(AX)\cap R^m_+$ and by Lemma 5.6 and Proposition 5.7, the extreme vectors $x_1,\dots,x_r$ of $\operatorname{Im} A\cap R^m_+$ satisfy the equivalent conditions of Lemma 5.6. From the equation $AXA=A$, we obtain $A^TX^TA^T=A^T$. Here $A^TX^T=(XA)^T$ is also a 0-symmetric idempotent with image space $A^T$. Likewise, $y_1,\dots,y_r$ satisfy the equivalent conditions of Lemma 5.6. ■

REMARK 5.10. The equivalences (ii)$\Leftrightarrow$(iv) and (ii)$\Leftrightarrow$(iii) are essentially contained in Jain [26, Corollary and Remark (1)].

THEOREM 5.11. *Let A be an $n\times n$ nonnegative matrix of rank r. The following are equivalent:*

(i) *$A^+$ is nonnegative and equals $A^\#$.*

(ii) *A is expressible as $\lambda_1x_{\sigma(1)}x_1^T+\cdots+\lambda_rx_{\sigma(r)}x_r^T$ where $x_i\in R^n_+$, $\sigma\in S(r)$, and $x_i^Tx_j=\delta_{ij}$. Then*

$$A^+=\frac{1}{\lambda_1}x_1x_{\sigma(1)}^T+\cdots+\frac{1}{\lambda_r}x_rx_{\sigma(r)}^T.$$

(iii) *There exists a permutation matrix P such that $PAP^T$ is a direct sum of* *matrices of the following three types (not necessarily all):*

(I) $uu^T$ *where* $u$ *is a positive vector,*

(II) *the form*

$$\begin{bmatrix} 0 & u_1u_2^T & 0 & \cdots & 0 \\ 0 & 0 & u_2u_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & u_{d-1}u_d^T \\ u_du_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

*where* $u_1,\ldots,u_d$ *are positive vectors, not necessarily of the same size,*

(III) *a zero matrix.*

*Proof.* (i)⇒(ii): Since $A^+=A^\#$, $\operatorname{Im} A^T=\operatorname{Im} A^+=\operatorname{Im} A^\#=\operatorname{Im} A$. Hence $\operatorname{Im} A\cap R^n_+=\operatorname{Im} A^T\cap R^n_+$, and in view of Theorem 4.3 and condition (ii) of Theorem 5.4, $A$ can be expressed in the required form.

(ii)⇒(i): It is easily verified that

$$A^+=A^\#=\frac{1}{\lambda_1}x_1x^T_{\sigma(1)}+\cdots+\frac{1}{\lambda_r}x_rx^T_{\sigma(r)}.$$

(ii)⇔(iii): Similar to the proof of Theorem 5.4, (ii)⇔(iii). Here we need the fact that the vectors $x_1,\ldots,x_r$ satisfy condition (iii) of Lemma 5.6. ■

REMARK 5.12. The equivalence of (i) and (iii) can be found in Jain, Goel, and Kwak [28, Theorem 2].

## 6. GREEN'S RELATIONS ON $N_n$

Let $S$ denote a semigroup and let $a,b\in S$. The relation $\mathcal{R}$ [$\mathcal{L}$, $\mathcal{J}$] is defined on $S$ by $a\,\mathcal{R}\,b$ [$a\,\mathcal{L}\,b$, $a\,\mathcal{J}\,b$] iff $a$ and $b$ generate the same principal right [left, two-sided] ideal in $S$. The relation $\mathcal{H}$ is defined to be $\mathcal{L}\cap\mathcal{R}$, and the relation $\mathcal{L}\mathcal{R}=\mathcal{R}\mathcal{L}$ is denoted by $\mathcal{D}$. These equivalence relations are known as the Green's relations, and they play a fundamental role in the study of the algebraic structure of semigroups. (For details, see Clifford and Preston [15, Chapter 2].)

If $S$ contains an identity element, as in the case of $N_n$ (the semigroup of $n\times n$ nonnegative matrices), the following equations define the Green's relations:

$$\begin{array}{llll}
a \mathcal{R} b & \Leftrightarrow & a=bx,\ b=ay & \text{for some } x, y \in S, \\
a \mathcal{L} b & \Leftrightarrow & a=xb,\ b=ya & \text{for some } x, y \in S, \\
a \mathcal{D} b & \Leftrightarrow & a \mathcal{R} c \text{ and } c \mathcal{L} b & \text{for some } c \in S, \\
a \mathcal{J} b & \Leftrightarrow & a=x_1 b x_2,\ b=y_1 a y_2 & \text{for some } x_1, x_2, y_1, y_2 \in S, \\
a \mathcal{H} b & \Leftrightarrow & a \mathcal{R} b \text{ and } a \mathcal{L} b. &
\end{array}$$

Following the notation of Burns, Fiedler, and Haynsworth [12], we denote by $G(A)$ the polyhedral cone generated by the column vectors of a matrix $A$. Consider the matrix equation $AX=B$, $X$ being nonnegative. Obviously, the equation says that each column of $B$ can be expressed as a nonnegative linear combination of the columns of $A$; equivalently $G(B) \subseteq G(A)$. So we readily have

THEOREM 6.1. *For $A, B \in N_n$, $A \mathcal{R} B$ iff $G(A)=G(B)$, and $A \mathcal{L} B$ iff $G(A^T)=G(B^T)$.*

Hartfiel, Maxson, and Plemmons [22] have provided characterizations of the $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{D}$ relations of $N_n$. Evidently they were aware of the significance of the cones generated by the column (or row) vectors of the concerned matrices, because the numbers of extreme vectors of these cones do appear in their characterizations. However, the above simple characterizations of the $\mathcal{R}$ and $\mathcal{L}$ relations do not appear in their paper. As the readers can verify, their characterization of the $\mathcal{R}$ relation follows readily from our Theorem 6.1. To illustrate the use of our characterizations of the $\mathcal{R}$ and $\mathcal{L}$ relations, we prove the following known result (Robinson [44, Theorem 5]).

PROPOSITION 6.2. *Let $E_r$ be an idempotent in $N_n$ of rank $r$ such that there exist permutation matrices $P$ and $Q$ with the property that*

$$PE_rQ=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

*Let $A$ be an element of $N_n$. Then $A \mathcal{H} E_r$ iff*

$$PAQ=\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

*where $A_1$ is an $r \times r$ monomial.*

*Proof.* If $A \mathcal{R} E_r$, then by Theorem 6.1, $G(A)=G(E_r)$. Hence $G(AQ)=G(A)=G(E_r)=G(E_rQ)$, and so $G(PAQ)=PG(AQ)$ (treat $P$ as a linear operator)$=PG(E_rQ)=G(PE_rQ)$. Thus $PAQ \mathcal{R} PE_rQ$. Similarly $A \mathcal{L} E_r$ implies that $PAQ \mathcal{L} PE_rQ$. It follows that $A \mathcal{H} E_r$ iff $PAQ \mathcal{H} PE_rQ$. So it is sufficient to prove the proposition for the case $P=Q=I_n$; that is, to prove that for any $A \in N_n$,

$$A \mathcal{H} E_r, \qquad \text{where} \quad E_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

iff

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{where} \quad A_1 \text{ is an } r \times r \text{ monomial.}$$

"If" part: It is obvious that $G(A)=G(E_r)$ and $G(A^T)=G(E_r^T)$. So $A \mathcal{R} E_r$ and $A \mathcal{L} E_r$. Hence $A \mathcal{H} E_r$.

"Only if" part: We first show that the last $n-r$ columns of $A=[a_{ij}]$ are zero. Assume the contrary. Then $a_{ij} \neq 0$ for some $i, j$, $r+1 \leqslant j \leqslant n$. It is easily seen that the $i$th row of $A$ cannot be expressed as a nonnegative linear combination of the rows of $E_r$. Hence $G(A^T) \not\subseteq G(E_r^T)$ and $A \not\mathcal{L} E_r$, which is a contradiction. Thus the nonzero columns of $A$ are among its first $r$ columns. Observe that the extreme vectors of the cone $G(E_r)$ are $e_1, \ldots, e_r$, where $e_i$ denotes the vector of $R^n$ with 1 in the $i$th position and 0 elsewhere. But these are also the extreme vectors of the cone $G(A)$ as $A \mathcal{R} E_r$. Hence these vectors, or their positive multiples, can be found among the first $r$ columns of $A$. It follows that $A$ is of the form

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A_1$ is $r \times r$ monomial. ∎

THEOREM 6.3. *Let $A, B \in N_n$. $A \mathcal{J} B$ iff there exists a subcone $K_1$ of $G(A)$ such that $K_1$ is mapped isomorphically onto $G(B)$ under the restriction of some $X_1 \in N_n$, and there exists a subcone $K_2$ of $G(B)$ such that $K_2$ is mapped isomorphically onto $G(A)$ under the restriction of some $X_2 \in N_n$.*

*Proof.* "Only if" part: Suppose $A \mathcal{J} B$. Then there exist $X_i, Y_i \in N_n$ ($i=1,2$) such that $X_1AY_1=B$ and $X_2BY_2=A$. The equations clearly imply rank $A=$ rank $B$, so $\dim G(A)=\dim G(B)$. The first equation says that $X_1$ maps the subcone $G(AY_1)$ of $G(A)$ onto $G(B)$. Since $\dim G(A)=\dim G(B)$, necessarily $\dim G(AY_1)=\dim G(B)$. Hence the restriction of $X_1$ maps $G(AY_1)$ isomorphically onto $G(B)$. Similarly, the subcone $G(BY_2)$ of $G(B)$ is mapped isomorphically onto $G(A)$ under the restriction of $X_2$.

"If" part: Denote by $b_i$ the $i$th column of $B$. By assumption, under the restriction of $X_1$, $K_1$ is mapped isomorphically onto $G(B)$. Hence for each $i$, there exists $c_i \in K_1$ such that $X_1 c_i = b_i$. Let $C$ be the $n\times n$ matrix $(c_1\ c_2\ \dots\ c_n)$. Then $X_1C=B$ and $G(C)=K_1$. As $K_1$ is a subcone of $G(A)$, there exists $Y_1 \in N_n$ such that $C=AY_1$. Hence $X_1AY_1=B$. Similarly there exists $Y_2\in N_2$ such that $X_2BY_2=A_1$. Therefore, $A \mathcal{J} B$. ■

It is well known that in any semigroup, $\mathcal{D}\subseteq\mathcal{J}$. Also, if a $\mathcal{D}$-class contains a regular element, then every element in the same $\mathcal{D}$-class is regular. For the semigroup $N_n$, we have the following stronger result, which is an improvement of Corollary 1 in [22].

Corollary 6.4. *Let $A, B\in N_n$. If $A$ is regular and $A \mathcal{J} B$, then $B$ is regular and $A \mathcal{D} B$.*

*Proof.* By Theorem 6.3, there exists a matrix $X\in N_n$ under the restriction of which some subcone $K$ of $G(B)$ is mapped isomorphically onto $G(A)$. Moreover, $\dim G(B)=\dim K=\dim G(A)$. Since $A$ is regular in $N_n$, $G(A)=\operatorname{Im} A\cap R^n_+$ is a simplicial cone. This, together with the inclusions $G(A)=XK\subseteq XG(B)\subseteq X(\operatorname{Im} B\cap R^n_+)\subseteq \operatorname{Im} A\cap R^n_+$ and the fact that $X|_{\operatorname{Im} B}$ is one-to-one, imply that the cones $K$, $G(B)$, and $\operatorname{Im} B\cap R^n_+$ coincide and are simplicial. Hence by Proposition 2.8, $B$ is regular. As $A$, $B$ are both regular and $A \mathcal{J} B$, by Corollary 2 of [22] we have $A\mathcal{D} B$. ■

We can also supply a more direct proof of the above result: if $C$, $D$ are nonnegative matrices such that their product $CD$ contains an $r\times r$ monomial submatrix, then so do $C$ and $D$; from this we can show that $B$ contains an $r\times r$ monomial submatrix, where $r=$ rank $B$.

Theorem 6.5. *Let $A, B\in N_n$. $A\mathcal{D} B$ iff there exist $X_1, X_2\in N_n$ such that $X_1G(A)=G(B)$, $X_2G(B)=G(A)$, and $X_2X_1|_{\operatorname{Im} A}$ is the identity map.*

*Proof.* "Only if" part: Suppose $A \mathcal{L} C$ and $C \mathcal{R} B$, $C\in N_n$. Then $X_1A=C$ and $X_2C=A$ for some $X_1, X_2\in N_n$, and $G(C)=G(B)$. Hence $X_1G(A)=G(B)$, $X_2G(B)=G(A)$. Moreover, $X_2X_1A=X_2C=A$, i.e., $X_2X_1|_{\operatorname{Im} A}$ is the identity map.

"If" part: Let $X_1, X_2 \in N_n$ such that $X_1 G(A) = G(B)$, $X_2 G(B) = G(A)$, and $X_2 X_1|_{\mathrm{Im}\,A}$ is the identity map. Clearly $X_1|_{\mathrm{Im}\,A}$, $X_2|_{\mathrm{Im}\,B}$ are one-to-one, and $X_2|_{\mathrm{Im}\,B} = (X_1|_{\mathrm{Im}\,A})^{-1}$. Let $c_1, \dots, c_r \in G(B)$ such that $X_2 c_i = a_i$, $1 \leq i \leq n$, where $a_i$ is the $i$th column of $A$. Denote by $C$ the matrix $[c_1 \ \cdots \ c_n]$. Then $X_2 C = A$. As $X_2|_{\mathrm{Im}\,B}$ is one-to-one, clearly $G(C) = G(B)$. Hence $C \mathcal{R} B$. From $X_2|_{\mathrm{Im}\,B} = (X_1|_{\mathrm{Im}\,A})^{-1}$, we obtain $X_1 A = C$. Thus $A \mathcal{L} C$. Therefore, $A \mathcal{D} B$. ■

The example given in Hartfiel, Maxson, and Plemmons [22] can be used to show that the condition "there exists $X_1 \in N_n$ under the restriction of which $G(A)$ is mapped isomorphically onto $G(B)$" alone is not sufficient for $A \mathcal{D} B$.

As shown in [22, Theorem 4], if $A \mathcal{D} B$ then $|\mathrm{Ext}\, G(A)| = |\mathrm{Ext}\, G(B)|$, where $|\mathrm{Ext}\, G(A)|$ denotes the number of extreme rays of the cone $G(A)$. (This is also clear from the above characterization of the $\mathcal{D}$ relation: for then, the cones $G(A)$ and $G(B)$ are linearly isomorphic.) In the example below we shall show that when $A \mathcal{J} B$, we do not necessarily have $|\mathrm{Ext}\, G(A)| = |\mathrm{Ext}\, G(B)|$. The construction of this example is suggested by our geometric characterization of the $\mathcal{J}$ relation.

EXAMPLE 6.6. Consider the nonnegative matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $G(B) \subseteq G(A)$, $|\mathrm{Ext}\, G(A)| = 3$, and $|\mathrm{Ext}\, G(B)| = 4$. Let $K_1$ be the subcone of $B$ generated by the vectors $(0,0,1,0)^T$, $(0,1,0,0)^T$, and $(1,2,0,0)^T$, and let

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Direct verification shows that $CK_1 = G(A)$. Hence, by Theorem 6.3, $A \mathcal{J} B$. Indeed, we have

$$B = A \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A = CB\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

## 7. REGULAR STOCHASTIC, COLUMN STOCHASTIC, AND DOUBLY STOCHASTIC MATRICES

If $S$ is a semigroup and $a$ is an element of $S$ for which there exists $x \in S$ satisfying $axa = a$, then $a$ is called a *regular* element of $S$. An $n \times n$ nonnegative matrix with a nonnegative (1)-inverse is clearly regular in the semigroup $N_n$. Regularity in $N_n$ was first studied by Plemmons [39].

The semigroups of $n \times n$ (row) stochastic matrices, column stochastic matrices, and doubly stochastic matrices will be denoted respectively by $S_n$, $T_n$, and $D_n$. Clearly a regular element of $S_n$, $T_n$, or $D_n$ is also regular in $N_n$, and so they can be treated by the same method which we have used in Section 5 for nonnegative matrices with various kinds of nonnegative generalized inverses. To obtain slightly more general results, we shall work with rectangular row or column stochastic matrices.

LEMMA 7.1. *Let $x_1, \ldots, x_r \in R^n_+$ satisfying the equivalent conditions of Lemma 5.1. The following are equivalent:*

(i) *There exists a stochastic idempotent with image space* $\operatorname{span}\{x_1, \ldots, x_r\}$.
(ii) $e = (1, \ldots, 1)^T \in \operatorname{span}\{x_1, \ldots, x_r\}$.
(iii) *There exist positive scalars $\lambda_i$ and an $n \times n$ permutation $P$ such that the vectors $P(\lambda_1 x_1), \ldots, P(\lambda_r x_r)$ are partitioned conformally:*

$$P(\lambda_1 x_1) = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline v_1 \end{bmatrix}, \ldots, \qquad P(\lambda_i x_i) = \begin{bmatrix} 0 \\ \hline \vdots \\ \hline 1 \\ \vdots \\ 1 \\ \hline \vdots \\ \hline 0 \\ \hline v_i \end{bmatrix}, \ldots, \qquad P(\lambda_r x_r) = \begin{bmatrix} 0 \\ \hline \vdots \\ \hline 0 \\ \hline 1 \\ \vdots \\ 1 \\ \hline v_r \end{bmatrix}.$$

*where $v_1,\dots,v_r$ are nonnegative vectors satisfying $v_{1k}+v_{2k}+\cdots+v_{rk}=1$ for each $k$, where $v_{ik}$ denotes the $k$th component of $v_i$, and may not appear.*

*Proof.* Since $\operatorname{span}\{x_1,\dots,x_r\}\cap R^n_+$ is an $r$-dimensional simplicial cone with $x_1,\dots,x_r$ as the extreme vectors, by Proposition 2.5 there is always a nonnegative idempotent $E$ with image space $\operatorname{span}\{x_1,\dots,x_r\}$. Furthermore $E$ is stochastic iff $Ee=e$ iff $e\in\operatorname{Im}E$. So the equivalence (i)$\Leftrightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii): By (ii), there exist scalars $\lambda_i$ such that $\lambda_1x_1+\cdots+\lambda_rx_r=(1,\dots,1)^T$. As the vectors $x_1,\dots,x_r$ satisfy the equivalent conditions of Lemma 5.1, for each $i$, $1\le i\le r$, there exists $k_i$ such that $x_{ik_i}\ne 0$, $x_{jk_i}=0$ for $j\ne i$. This forces each $\lambda_i$ to be positive. By condition (iii) of Lemma 5.1, for some permutation matrix $P$,

$$P(\lambda_1x_1)=\begin{bmatrix}u_1\\0\\\vdots\\0\\v_1\end{bmatrix},\dots,\qquad P(\lambda_rx_r)=\begin{bmatrix}0\\\vdots\\0\\u_r\\v_r\end{bmatrix},$$

where the vectors $u_i$, $v_i$ satisfy the conditions given there. Thus

$$e=Pe=\begin{bmatrix}u_1\\\vdots\\u_r\\\sum\limits_{i=1}^{r}v_i\end{bmatrix}.$$

Hence each vector $u_i$ is a vector with all entries 1, and $\sum_{i=1}^r v_{ij}=1$ for each $j$.

(iii)$\Rightarrow$(ii): Obvious. ■

By Lemmas 5.6 and 7.1 and Proposition 5.7 we readily obtain

LEMMA 7.2. *Let $x_1,\dots,x_r\in R^n_+$. The following are equivalent:*

(i) $\operatorname{span}\{x_1,\dots,x_r\}\cap R^n_+$ *is an $r$-dimensional simplicial cone with extreme vectors $x_1,\dots,x_r$, and there exists a doubly stochastic idempotent with image space* $\operatorname{span}\{x_1,\dots,x_r\}$.

(ii) *In each of the vectors $x_1,\dots,x_r$, the nonzero entries are the same, no two of these vectors have nonzero entries at the same position, and there is no zero position common to all.*

Using Lemma 7.2, we now give a new proof of the well-known characterization of the maximal subgroups of $D_n$. The result was obtained previously by Schwarz [48], Farahat [18], and Wall [56].

PROPOSITION 7.3. *Every maximal subgroup of $D_n$ is isomorphic to a finite direct product of full symmetric groups.*

*Proof.* Let $\mathcal{G}$ be a maximal subgroup of $D_n$ with $E$ as the identity. Let $x_1,\dots,x_r$ be the extreme probability vectors of the simplicial cone $\operatorname{Im} E\cap R^n_+$. These vectors clearly satisfy the equivalent conditions of Lemma 7.2. Let $n_i$ be the number of nonzero entries in $x_i$. Then each nonzero entry of $x_i$ is $1/n_i$. Also $n_1+n_2+\cdots+n_r=n$. Let $A\in\mathcal{G}$. Since $A$ is column stochastic and has a nonnegative group inverse, $A$ permutes the extreme probability vectors $x_1,\dots,x_r$ among themselves. Note that $n_1x_1+n_2x_2+\cdots+n_kx_k=e$. As $A$ is also row stochastic,

$$e=Ae=n_1Ax_1+n_2Ax_2+\cdots+n_kAx_k.$$

This, together with the fact that $x_1,\dots,x_r$ satisfy Lemma 7.2, forces $n_i=n_j$ whenever $Ax_i=x_j$. Thus $A$ permutes those $x_i$ with the same number of nonzero entries among themselves. The rest of the proof is fairly obvious. ∎

PROPOSITION 7.4. *Let $A$ be an $m\times n$ stochastic matrix of rank $r$. Then $A$ has a nonnegative* (1)*-inverse iff $A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$, where the vectors $x_1,\dots,x_r\in R^m_+$ satisfy the equivalent conditions of Lemma* 7.1, *and $y_1,\dots,y_r\in R^n_+$ satisfy the equivalent conditions of Lemma* 5.1.

*Proof.* The "if" part obviously follows from Theorem 5.2. To prove the "only if" part, suppose $A$ has a nonnegative (1)-inverse. Then $A=x_1y_1^T+\cdots+x_ry_r^T$, where $x_i\in R^m_+$, $y_i\in R^n_+$ and both families of vectors $x_1,\dots,x_r$ and $y_1,\dots,y_r$ satisfy the equivalent conditions of Lemma 5.1. Moreover, $x_1,\dots,x_r$ are extreme vectors of the simplicial cone $\operatorname{Im} A\cap R^m_+$. Since $A$ is stochastic, the vector $e=(1,\dots,1)^T\in\operatorname{Im} A$. Hence any $n\times n$ nonnegative idempotent with the same image space as $A$ is stochastic. As $A$ has a nonnegative (1)-inverse, we know there is one such idempotent. Therefore, the vectors $x_1,\dots,x_r$ satisfy the equivalent conditions of Lemma 7.1. ∎

By taking transposes, we obtain

PROPOSITION 7.5. *Let $A$ be an $m\times n$ column stochastic matrix of rank $r$. Then $A$ has a nonnegative* (1)*-inverse iff $A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$,* *where the vectors* $x_1,\dots,x_r \in R^m_+$ *satisfy the equivalent conditions of Lemma* 5.1 *and the vectors* $y_1,\dots,y_r \in R^n_+$ *satisfy the equivalent conditions of Lemma* 7.1.

THEOREM 7.6. *Let $A$ be an $m\times n$ stochastic matrix of rank $r$. The following are equivalent:*

(i) *$A$ has a stochastic* (1)*-inverse.*

(ii) *$A$ is expressible in the form $x_1y_1^T+\cdots+x_ry_r^T$, where $x_i,\dots,x_r\in R^m_+$ satisfy the equivalent conditions of Lemma* 7.1 *and $y_1,\dots,y_r\in R^n_+$ satisfy the equivalent conditions of Lemma* 5.6.

(iii) *There exist permutation matrices $P$ and $Q$ such that the matrix $PAQ$ has the following block form:*

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & A_r & 0 \\ U_1 & U_2 & \cdots & U_r & 0 \end{bmatrix},$$

*in which each $A_i$ is a positive stochastic matrix of rank* 1 *and $U_i=D_iA_i'$, where $D_i$ is a nonnegative diagonal matrix, $D_1+\cdots+D_r=I$, and $A_i'$ consists of rows of $A_i$.*

(iv) *$A$ has a nonnegative* (1)*-inverse, and each column of $A$ is either an extreme vector of the cone* $\operatorname{Im} A\cap R^m_+=G(A)$ *or a zero column.*

*Proof.* (i)$\Rightarrow$(ii): By Proposition 7.4, $A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$, where $x_1,\dots,x_r\in R^m_+$ satisfy the equivalent conditions of Lemma 7.1, and $y_1,\dots,y_r\in R^n_+$ are extreme vectors of the simplicial cone $\operatorname{Im} A^T\cap R^n_+$. As $A$ has a stochastic (1)-inverse, $A^T$ has a column stochastic (1)-inverse. So there exists a column stochastic idempotent with image space $\operatorname{Im} A^T$. Therefore, by Proposition 5.7 and Lemma 5.6, $y_1,\dots,y_r$ satisfy the equivalent conditions of Lemma 5.6.

(ii)$\Rightarrow$(i): Suppose $A$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$, where the vectors $x_i, y_i$ satisfy the given conditions. We may assume that $y_1,\dots,y_r$ are probability vectors. By condition (ii) of Lemma 5.1, for each $i$ there exists $k_i$ such that $x_{ik_i}\neq 0$ and $x_{jk_i}=0$ for $j\neq i$. For any such $k_i$, the $k_i$th row of $A$ is clearly $x_{ik_i}y_i^T$. But $A$ is stochastic and $y_i$ is chosen to be a probability vector, so $x_{ik_i}=1$. Now using Lemma 7.1, it is not difficult to see that, for some permutation matrix $P$,

$$(Px_i)^T = \left[ 0 \mid \cdots \mid 1 \quad \cdots \quad 1 \mid \cdots \mid 0 \mid v_i^T \right], \qquad 1 \leqslant i \leqslant r,$$

where $v_1, \dots, v_r$ are nonnegative vectors of the same size, say $l_{r+1}$, such that, for each $k$, $v_{1k} + \cdots + v_{rk} = 1$ and there exist at least two different $i$ satisfying $v_{ik} \neq 0$. Let the number of 1's in the entries of $x_i$ be $l_i$. Let $b \in R^m_+$ be given by

$$b_{ik} = \begin{cases} \dfrac{1}{l_i} & \text{whenever} \quad x_{ik} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly each $b_i$ is a probability vector and $b_i^T x_j = \delta_{ij}$. Next let $a_i \in R^n_+$ be given by

$$a_{ik} = \begin{cases} 1 & \text{if} \quad y_{ik} \neq 0, \\ \dfrac{1}{r} & \text{if} \quad y_{jk} = 0 \quad \text{for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

Again, we have, $a_i^T y_j = \delta_{ij}$. So, by condition (ii) of Theorem 5.2, $a_1 b_1^T + \cdots + a_r b_r^T$ is a nonnegative semiinverse of $A$. We leave it to the reader to verify that this matrix is in fact stochastic.

(ii)$\Leftrightarrow$(iii): Similar to the proof of Theorem 5.2, (ii)$\Leftrightarrow$(iii).

(ii)$\Rightarrow$(iv): Follows readily from the fact that the vectors $y_1, \dots, y_r$ satisfy condition (iii) of Lemma 5.6.

(iv)$\Rightarrow$(ii): As $A$ has a nonnegative (1)-inverse, $A$ has a "unique" representation as a sum of $r$ rank-one nonnegative matrices: $A = x_1 y_1^T + \cdots + x_r y_r^T$. The vectors $x_1, \dots, x_r$ are the extreme vectors of the simplicial cone $\operatorname{Im} A \cap R^m_+$, and the vectors $y_1, \dots, y_r$ satisfy the equivalent conditions of Lemma 5.1. Note that no two of the vectors $y_1, \dots, y_r$ have nonzero entries at the same position; otherwise, the corresponding column of $A$, being a sum of distinct extreme vectors of $\operatorname{Im} A \cap R^m_+$, would not be an extreme vector of $\operatorname{Im} A \cap R^m_+$. So the vectors $y_1, \dots, y_r$ satisfy the equivalent conditions of Lemma 5.6. ∎

Remark 7.7. The equivalence of (i)$\Leftrightarrow$(iii) in the above theorem can be found in Wall [55, Theorem 1].

We leave it to the reader to formulate the corresponding theorem for column stochastic matrices. Using Theorem 7.6 and the results of Section 5, we can also obtain characterizations of stochastic matrices with the stochastic group inverse, Moore-Penrose inverse, etc., but we omit the details.

We now use Theorem 7.6 to study the Green's relations on $S_n$ for regular elements. Wall [56] has studied the problem before. Our approach allows us to obtain the transparent characterization below. The readers can verify that Theorems 2.1 and 3.1 of Wall [56] follow readily from our results. Before coming to our characterization, note that for $A, B \in S_n$, $A \mathcal{L} B$ in $S_n$ if $A \mathcal{L} B$ in $N_n$.

THEOREM 7.8. *Let $A, B$ be regular elements of $S_n$. Then $A \mathcal{R} B$ in $S_n$ iff $A \mathcal{R} B$ in $N_n$.*

*Proof.* It is necessary to consider only the "if" part. Assume $A \mathcal{R} B$ in $N_n$. Then $\operatorname{Im} A \cap R^n_+ = G(A) = G(B) = \operatorname{Im} B \cap R^n_+$. Let $x_1, \dots, x_r$ be the extreme vectors of $\operatorname{Im} A \cap R^n_+$, $r = \operatorname{rank} A$. Since $A$ is regular in $S_n$, there is a stochastic idempotent with image space $\operatorname{Im} A$. So the vectors $x_1, \dots, x_r$ satisfy the equivalent conditions of Lemma 7.1. Replacing $x_i$ by suitable positive multiples, if necessary, we may assume that for some permutation matrix $P$,

$$(Px_i)^T = \left[0 \mid \cdots \mid 1 \quad \cdots \quad 1 \mid \cdots \mid 0 \mid v_i^T\right], \qquad 1 \leq i \leq r,$$

where $v_i$ are nonnegative vectors satisfying the requirements given in Lemma 7.1, condition (iii). By Theorem 7.6,

$$A = x_1 y_1^T + \cdots + x_r y_r^T,$$

$$B = x_1 w_1^T + \cdots + x_r w_r^T,$$

where $y_1, \dots, y_r$ (also $w_1, \dots, w_r$) satisfy the equivalent conditions of Lemma 5.6. By our assumption on the $x_i$ and the fact that $A$ (also $B$) is stochastic, $y_1, \dots, y_r$ (also $w_1, \dots, w_r$) are probability vectors [cf. the proof of Theorem 7.6, (ii)$\Rightarrow$(i)]. Now let $a_1, \dots, a_r \in R^n_+$ be given by

$$a_{ik} = \begin{cases} 1 & \text{if} \quad y_{ik} \neq 0 \text{ and } y_{jk} = 0 \text{ for } j \neq k, \\ \dfrac{1}{r} & \text{if} \quad y_{jk} = 0 \quad \text{for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $a_1, \dots, a_r$ satisfy $y_i^T a_j = \delta_{ij}$, as well as the equivalent conditions of

Lemma 7.1. Hence by Theorem 7.6 $X=a_1w_1^T+\cdots+a_rw_r^T\in S_n$, and

$$
\begin{aligned}
AX&=(x_1y_1^T+\cdots+x_ry_r^T)(a_1w_1^T+\cdots+a_rw_r^T)\\
&=x_1w_1^T+\cdots+x_rw_r^T\\
&=B.
\end{aligned}
$$

Similarly, there exists $Y\in S_n$ such that $BY=A$. Therefore $A\mathcal{R}B$ in $S_n$. ■

COROLLARY 7.9. *For regular elements* $A$, $B$ *of* $S_n$, $A\mathcal{H}B$ (*or* $A\mathcal{D}B$) *in* $S_n$ *iff* $A\mathcal{H}B$ (*or* $A\mathcal{D}B$) *in* $N_n$.

Clearly we also have the corresponding results for the Green's relations on $T_n$ for regular elements.

The characterization of the $\mathcal{R}$ relation of $S_n$ is still an unsettled question. Of course, it is equivalent to characterizing the $\mathcal{L}$ relation of $T_n$. A more general question is to consider the single matrix equation $XB=A$ over the semigroup $T_n$. This problem is closely related to an old (wrong) conjecture of Kakutani on doubly stochastic matrices to be described below.

Let $x=(x_1,\ldots,x_n)^T$, $y=(y_1,\ldots,y_n)^T$ be vectors of $R^n$. Denote by $x_1^*,\ldots,x_n^*$ the numbers $x_1,\ldots,x_n$ arranged in nonincreasing order, and let $y_1^*,\ldots,y_n^*$ be defined analogously. If the relations

$$y_1^*+\cdots+y_k^*\leqslant x_1^*+\cdots+x_k^*,\qquad 1\leqslant k\leqslant n,$$

and $y_1+\cdots+y_n=x_1+\cdots+x_n$ are satisfied, we shall write $y\prec x$. Hardy, Littlewood, and Pólya [21, Theorem 46] obtained the following fundamental result (for further references, see Mirsky [35]):

For $x,y\in R^n$, $y\prec x$ iff $y=Nx$ for some $N\in D_n$.

It is straightforward to show that the relation $\prec$ is reflexive and transitive. S. Kakutani conjectured that if $A,B\in D_n$ and $Ax\prec Bx$ for every vector $x\in R^n$, then there exists a matrix $X\in D_n$ such that $A=XB$. This conjecture was shown (Sherman [50]) to be generally incorrect for $n\geqslant 4$, but Schreiber [46] proved that it is valid when $B$ is nonsingular. Mirsky [35, Problem (vi) at the end] asked for a convenient condition which, in the general case, would ensure the existence of the required matrix $X$. We suggest that attention should be directed to the corresponding problem for column stochastic matrices, in view of the following:

PROPOSITION 7.10. *Let $A$, $B$ be real matrices of order $m\times p$ and $n\times p$ respectively. The following are equivalent*:

(i) $A=NB$ *for some $m\times n$ nonnegative matrix $N$.*

(ii) *For each $x\in R^p$, there exists an $m\times n$ nonnegative matrix $N_x$ such that $Ax=N_xBx$.*

(iii) *For any $x\in R^p$, $Bx\geqslant 0$ implies $Ax\geqslant 0$. (Here $y\geqslant 0$ means the vector $y$ is entrywise nonnegative.)*

*Proof.* The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are obvious. To establish (iii)$\Rightarrow$(i), observe that condition (i) really says that each row vector of $A$ can be expressed as a nonnegative linear combination of the row vectors of $B$, or equivalently, $G(A^T)\subseteq G(B^T)$. Hence, it is sufficient to show that $G(B^T)^*\subseteq G(A^T)^*$. (Recall that $K^*$ denotes the dual cone of $K$.) Let $x\in G(B^T)^*$. Then $Bx=(b_1^Tx,\dots,b_n^Tx)^T\geqslant 0$, where $b_i^T$ denotes the $i$th row vector of $B$. By (iii), this implies $Ax\geqslant 0$, so $x\in G(A^T)^*$. ■

COROLLARY 7.11. *For $A, B\in S_n$, $A=XB$ for some $X\in S_n$ iff for every $x\in R^n$, $Ax=X_xBx$ for some $X_x\in S_n$.*

*Proof.* "Only if" part: Obvious.

"If" part: By Proposition 7.10, there exists $X\in N_n$ such that $A=XB$. Since $A$, $B$ are stochastic, $e=Ae=XBe=Xe$. Therefore, $X\in S_n$. ■

The example of Sherman [50] also shows that for $A, B\in T_n$, the condition "for every $x\in R^n$ there exists some $X_x\in T_n$ such that $Ax=X_xBx$" does not imply the existence of some $X\in T_n$ satisfying $A=XB$. A relevant question is the following: Let $A, B\in T_n$ such that $A=XB$ for some $X\in N_n$. When will there be a matrix $Y\in T_n$ such that $A=YB$? The following is a partial answer.

PROPOSITION 7.12. *Let $A, B\in T_n$ (respectively $D_n$). Suppose that $A=XB$ for some $X\in N_n$. If there exists a column stochastic idempotent with the same image space as $B$ (for instance, when $B$ is regular in $T_n$ or is nonsingular), then there exists $Y\in T_n$ (respectively $D_n$) such that $A=YB$.*

*Proof.* The assertion for the $D_n$ case follows readily from the $T_n$ case: if $A, B\in D_n$ and $Y\in T_n$ such that $A=YB$, then $e=Ae=YBe=Ye$ and so $Y\in D_n$.

Denote by $H_e$ the hyperplane $\{x\in R^n : e^Tx=1\}$. Recall that a nonnegative matrix $Z\in T_n$ iff $Z(H_e)\subseteq H_e$. We claim that $X(H_e\cap \operatorname{Im} B)\subseteq H_e$. Let $y\in H_e\cap \operatorname{Im} B$. Then $y=Bx$ for some vector $x$. Since $1=e^Ty=e^TBx=e^Tx$, $x\in H_e$. Hence $Xy=XBx=Ax\in H_e$, as $A\in T_n$. Let $P$ be a column stochastic idempotent with image space $\operatorname{Im} B$. Then $XP\in\pi(R^n_+)$ and $XP(H_e)\subseteq X(H_e\cap \operatorname{Im} B)\subseteq H_e$. Moreover, $(XP)B=X(PB)=XB=A$. So $Y=XP$ is the required column stochastic operator. ■

Corollary 7.13. *Let $A, B\in T_n$ such that $A\,\mathfrak{L}\,B$ in $N_n$. If each of* $\operatorname{Im} A$ *and* $\operatorname{Im} B$ *is the image space of an idempotent of $T_n$, then $A\,\mathfrak{L}\,B$ in $T_n$.*

Corollary 7.14. *Let $A, B$ be regular elements of $T_n$. Then $A\,\mathfrak{L}\,B$ in $T_n$ iff $A\,\mathfrak{L}\,B$ in $N_n$.*

So we also have an alternative proof of Theorem 7.8.

The question which is raised after Corollary 7.11 probably does not have any nice solution, in view of the following:

Proposition 7.15. *Let $B$ be an element of $T_n$ which is regular in $N_n$ but not in $T_n$. There always exists a regular element $A$ of $T_n$ such that $A\,\mathfrak{L}\,B$ in $N_n$, but $A\,\not\!\mathfrak{L}\,B$ in $T_n$. Then there exists $X\in N_n$ such that $A=XB$, but there does not exist $Y\in T_n$ such that $A=YB$.*

*Proof.* By Proposition 7.5 and the "dual" of Theorem 7.6, $B$ is expressible as $x_1y_1^T+\cdots+x_ry_r^T$, where $x_1,\ldots,x_r\in R^n_+$ satisfy the equivalent conditions of Lemma 5.1 but not the conditions of Lemma 5.6, and $y_1,\ldots,y_r\in R^n_+$ satisfy the equivalent conditions of Lemma 7.1. We may choose $x_1,\ldots,x_r$ to be probability vectors. Let $a_1,\ldots,a_r$ be any probability vectors of $R^n_+$ that satisfy the conditions of Lemma 5.6, and let $A=a_1y_1^T+\cdots+a_ry_r^T$. Then $A$ is regular in $T_n$. Furthermore, $A\,\mathfrak{L}\,B$ in $N_n$ because $G(A^T)=G(B^T)$ (both cones are generated by $y_1,\ldots,y_r$). But $A\,\not\!\mathfrak{L}\,B$ in $T_n$; otherwise $B$ is regular in $T_n$. Since $A\,\mathfrak{L}\,B$ in $N_n$, there exist $X, U\in N_n$ such that $A=XB$ and $B=UA$. As $A$ is also regular in $T_n$, by Proposition 7.12 there exists $V\in T_n$ such that $B=VA$. Hence there does not exist $Y\in T_n$ satisfying $A=YB$, lest $A\,\mathfrak{L}\,B$ in $T_n$ ■

Example 7.16. Let

$$A=\begin{bmatrix} \frac14 & \frac14 & \frac1{12} & \frac1{12}\\ \frac12 & \frac12 & \frac16 & \frac16\\ 0 & 0 & \frac14 & \frac14\\ \frac14 & \frac14 & \frac12 & \frac12 \end{bmatrix},\qquad B=\begin{bmatrix} \frac12 & \frac12 & \frac16 & \frac16\\ \frac12 & \frac12 & \frac16 & \frac16\\ 0 & 0 & \frac13 & \frac13\\ 0 & 0 & \frac13 & \frac13 \end{bmatrix}.$$

Obviously both $A, B \in T_4$. Furthermore, both cones $G(A^T)$ and $G(B^T)$ are generated by the vectors $(3,3,1,1)^T$ and $(0,0,1,1)^T$. So $A \mathcal{L} B$ in $N_4$. Observe that $\operatorname{Im} B \cap R^4_+$ is the two-dimensional cone generated by the vectors $(1,1,0,0)^T$ and $(0,0,1,1)^T$. By Lemma 5.6 and Proposition 5.7, there exists a column idempotent with image space $\operatorname{Im} B$, and by Proposition 7.12 there exists $Y \in T_4$ such that $YB = A$. Indeed, we may choose

$$Y = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{5}{8} & \frac{5}{8} \end{bmatrix}.$$

Note that the matrix

$$W = \begin{bmatrix} -\frac{3}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{3}{2} \\ 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 \end{bmatrix}$$

satisfies $WA = B$. Furthermore, $W$ belongs to the semigroup of all $4 \times 4$ real matrices with columns sums 1. (The intersection of this semigroup with $N_4$ gives $T_4$.) Hence, we also have $A \mathcal{L} B$ in this semigroup. However, there does not exist $X \in T_4$ such that $XA = B$, because for any such $X = (x_{ij})$, we have

$$\begin{pmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix} A = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

whence $x_{31} = x_{32} = x_{34} = 0$ and $x_{33} = \frac{4}{3}$, which is a contradiction. So we do not have $A \mathcal{L} B$ in $T_4$.

Montague and Plemmons [37] have considered the solvability of the system of matrix equations

$$AX = B \quad \text{and} \quad BY = A$$

where $A, B$ are $m \times n$ real matrices and $X, Y \in D_n$. It is found that the equations are solvable iff $A = BP$ for some permutation matrix $P$. Then the result is used to give the characterizations of the Green's relations on the semigroup $D_n$. This result is clearly also equivalent to the following: the system of matrix equations $XA = B$ and $YB = A$, where $A, B$ are $m \times n$ real matrices, has solutions $X, Y$ in $D_n$ iff there exists a permutation matrix $P$ such that $A=PB$. Whereas Kakutani's conjecture is wrong, it might be interesting to note that we have the following related result, which obviously extends the above result of Montague and Plemmons. (The short proof of Montague and Plemmons's result given in Berman and Plemmons [9, Theorem 5.1] is far from complete.)

PROPOSITION 7.17. *Let $A, B$ be $m\times n$ real matrices. Suppose that for each $x\in R^n$, $Ax\prec Bx$ and $Bx\prec Ax$. Then there exists an $m\times m$ permutation matrix $P$ such that $A=PB$.*

*Proof.* In the usual notation we write $Ax\approx Bx$ for $Ax\prec Bx$ and $Bx\prec Ax$. By Hardy, Littlewood, and Polya's result, the given assumption readily implies that for each $x\in R^n$, there exists an $m\times m$ permutation matrix $P_x$ such that $Ax=P_xBx$ (see Montague and Plemmons [37, Lemma 2]); in other words, each component of $Ax$ is a component of $Bx$ with the same multiplicity, and conversely. Our contention is: the matrices $A$ and $B$ have the same set of row vectors, and the multiplicities of all row vectors in $A$ and $B$ are the same. Denote the $i$th row vector of $A$ ($B$) by $a_i^T$ ($b_i^T$). Choose a vector $y\in R^n$ such that

$$y\notin \bigcup_{\substack{1\leqslant i,j\leqslant n\\ b_i\neq a_j}} (\operatorname{span}(b_i-a_j))^{\perp}.$$

(The case $b_1=\cdots=b_m=a_1=\cdots a_m$ is trivial.) Obviously, $b_i\neq a_j$ iff $b_i^Ty\neq a_j^Ty$. But by assumption $Ay\approx By$, i.e. $(a_1^Ty,\dots,a_m^Ty)^T\approx(b_1^Ty,\dots,b_m^Ty)^T$. Hence, our contention follows readily. ■

REMARK 7.18. The reader may observe that in the above proof we need the elementary fact that the union of a finite number of proper subspaces of a linear space cannot be the linear space itself. For that matter, we can replace the assumption "for each $x\in R^n$" by weaker ones, for instance, by "for each probability vector $x$."

## 8. FINAL REMARKS

(1) Following the work of Richman and Schneider [43] and Borosh, Hartfiel and Maxson [10] on the prime elements of $N_n$, the author [53] has made a study of the factorization problem in $N_n$ as well as in $\mathfrak{B}_n$, the

semigroup of $n\times n$ Boolean relation matrices, and their relationship. Here we are involved with again the cones associated with the concerned matrices, and the subspaces of $\mathcal{V}_n$, the Boolean space of dimension $n$ (for reference, see Plemmons [38]).

(2) Like Kakutani, we may ask the following question for any semigroup S of $n\times n$ real matrices which contains the identity matrix:

Let $A, B\in S$, and suppose that for every $x\in R^n$, $Ax=N_xBx$ for some $N_x\in S$. Does this imply the existence of some $N\in S$ such that $A=NB$?

We have verified that the answer is in the affirmative if S is one of the following semigroups: $n\times n$ real matrices, $N_n$, $S_n$, $n\times n$ real matrices with constant column sums 1, and $n\times n$ real matrices with constant row sums 1. The answer is in the negative for the semigroups $T_n$ and $D_n$.

(3) There is evidence which suggests that the geometry of the nonnegative orthant is a subject worthwhile to study. Its results are often fundamental to unified treatments of diverse topics. Ben-Israel [3] shows that the main results in the theory of linear inequalities in finite-dimensional vector spaces follow from an elementary property of the intersections of the nonnegative orthant with pairs of complementary orthogonal subspaces. Saunders and Schneider [45] demonstrate the use of the classical Gordon-Stiemke theorem in combinatorial matrix theory. In this paper, based on results about subspaces which are the image spaces of different kinds of nonnegative idempotents (Lemmas 5.1, 5.6, 7.1, 7.2; Propositions 2.5, 5.7), we have offered a unified treatment on different topics that are concerned with nonnegative matrices having nonnegative (1)-inverses. Our method certainly has its own limitations. However, for a further study of the algebraic properties of nonnegative matrices, the geometry of the nonnegative orthant will probably still play a useful role. Before we end, we pose the following problems which arise from our study of the Green's relations on $N_n$.

Problem 1. Let $A, B\in N_n$. Suppose that under the restriction of some $X\in N_n$, the cones $G(A)$ and $\operatorname{Im} A\cap R^n_+$ are mapped isomorphically onto the cones $G(B)$ and $\operatorname{Im} B\cap R^n_+$ respectively. Does this imply $A\mathcal{D}B$? (The converse is true.)

Problem 2. Let $H_1, H_2$ be subspaces of $R^n$ such that the cones $H_1\cap R^n_+$ and $H_2\cap R^n_+$ are linearly isomorphic. Does there always exist some $X\in N_n$ which takes one cone onto the other?

We have proved that the result is true if both cones are of dimension $n-1$. The problem is clearly some kind of extension problem (cf. Kelley and Namioka [33, Theorem 3.3], a geometric version of Hahn-Banach theorem). Any results in that direction will probably be useful.

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